Discrete Mathematics 339 (2016) 2857–2860

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

On the multicolor Ramsey number of a graph with m edges

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ARTICLE INFO

Article history:
Received 27 July 2014
Received in revised form 15 February 2016
Accepted 18 May 2016
Available online 1 July 2016

Keywords:
Multicolor Ramsey number

ABSTRACT

The multicolor Ramsey number $r_k(F)$ of a graph F is the least integer n such that in every coloring of the edges of K_n by k colors there is a monochromatic copy of F. In this short note we prove an upper bound on $r_k(F)$ for a graph F with m edges and no isolated vertices of the form $k^{6km^{2/3}}$ addressing a question of Sudakov (2011). Furthermore, the constant in the exponent in the case of bipartite F and two colors is lowered so that $r_2(F) \le 2^{(1+o(1))2\sqrt{2m}}$ improving the result of Alon, Krivelevich and Sudakov (2003).

1. Introduction

The by now classical theorem of Ramsey [11] states that no matter how one colors the edges of the large enough complete graph K_n with two colors, say red and blue, there will always be a monochromatic copy of K_t in it. The smallest such n is called the Ramsey number, denoted by $r(t)$ or $r(K_t)$. First lower and upper bounds on $r(t)$ were obtained by Erdős and Szekeres [8]: $r(t) \le \binom{2t-2}{t-1}$ and by Erdős [6]: $r(t) \ge 2^{t/2}$. Despite numerous efforts by various researchers, the best lower and upper bounds remain asymptotically $2^{(1+o(1))t/2}$ and $2^{(1+o(1))2t}$, for the currently best bounds see Conlon [5] and Spencer [12].

Thus, one turned to the study of Ramsey numbers of graphs other than complete graphs K_t. The multicolor Ramsey number for k colors of a graph F, denoted $r_k(F)$, is defined as the smallest number n such that in any coloring of $E(K_n)$ by k colors there is a monochromatic copy of a graph F in one of the k colors. Much attention was drawn by the conjectures of Burr and Erdős [2] about Ramsey numbers of graphs F of bounded maximum degree or degeneracy. These conjectures stated that such Ramsey numbers for F are linear in the number of vertices of F. The first conjecture has been resolved positively by Chvatál, Rödl, Szemerédi and Trotter [4] and the latter was recently proved by Lee [10].

A related conjecture of Erdős and Graham [7] states that among all graphs F with $m = \binom{t}{2}$ edges and no isolated vertices the Ramsey number $r(t)$ of the complete graph K_t is an upper bound on $r_2(F)$. A relaxation conjectured by Erdős [3] states that at least $r(F) \le 2^{c\sqrt{m}}$ should hold for any graph F with m edges and no isolated vertices and some absolute constant c. This was verified by Alon, Krivelevich and Sudakov [1] who showed that if F is bipartite, has m edges and no isolated vertices then $r(F) \le 2^{16\sqrt{m}+1}$, and for nonbipartite F showing $r(F) \le 2^{7\sqrt{m}\log_2 m}$. Finally, the general case was settled by Sudakov [14] who proved $r(F) \le 2^{250\sqrt{m}}$.

In his concluding remarks in [14], Sudakov mentions that the methods used to settle the general case are not extendible to more colors and it would be interesting to understand the growth of $r_k(F)$. It is easy to see that there is an upper bound on $r_k(F)$ of the form k^{2km} by finding a monochromatic copy of $K_{2m} \supset F$ using the classical color focussing argument. In this note we prove to the best of our knowledge a first nontrivial upper bound $r_k(F) \le k^{6km^{2/3}}$.

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http://dx.doi.org/10.1016/j.disc.2016.05.021
0012-365X/

Theorem 1. *Let F be a graph with m edges and no isolated vertices. Then, for $k \geq 3$ it holds*

$$r_k(F) \leq k^{3\cdot 2^{-1/3}km^{2/3}+k(2m)^{1/3}} 8m.$$

Further we study the case when F is bipartite and show an upper bound $r_k(F) \leq k^{(1+o(1))2\sqrt{mk}}$.

Theorem 2. *Let F be a bipartite graph with m edges and no isolated vertices. Then, for $k \geq 2$ it holds*

$$r_k(F) \leq 2^6 m^{3/2} k^{2\sqrt{km}+1/2}.$$

Note that in the case $k = 2$, Theorem 2 is an improvement of the above mentioned result of Alon, Krivelevich and Sudakov [1] to $r(F) \leq 2^{(1+o(1))2\sqrt{2m}}$.

The methods we use are slight modifications of the arguments of Fox and Sudakov from [9] and of Alon, Krivelevich and Sudakov [1]. The paper is organized as follows. In Section 2 we collect some results and observations used in our proofs, in Section 3 we prove Theorem 2 and in Section 4 we show Theorem 1.

2. Some auxiliary results

Here we collect several results from [9] and one small graph theoretic estimate. The first prominent lemma we use is the so-called dependent random choice lemma, stating that in a bipartite dense graph one finds a large vertex subset in one class, most of whose d-tuples have many common neighbors on the other side.

Lemma 3 (*Dependent Random Choice, Lemma 2.1[9]*)**.** *If $\varepsilon > 0$ and $G = (V_1, V_2; E)$ is a bipartite graph with $|V_1| = |V_2| = N$ and at least εN^2 edges, then for all positive integers a, d, t, x, there is a subset $A \subset V_2$ with $|A| \geq 2^{-\frac{1}{a}} \varepsilon^t N$, such that for all but at most $2\varepsilon^{-ta} \left(\frac{x}{N}\right)^t \left(\frac{|A|}{N}\right)^a \binom{N}{d}$ d-sets S in A, we have $|N(S)| \geq x$.*

The following lemma allows one to embed a graph H with bounded degree and bounded chromatic number into a dense graph G given along with a nested sequence of sets, where the parts of H are supposed to be embedded into.

Lemma 4 (*Lemma 4.2 in [9]*)**.** *Suppose G is a graph with vertex set V_1, and let $V_1 \supset \cdots \supset V_q$ be a family of nested subsets of V_1 such that $|V_q| \geq x \geq 4n$, and for $1 \leq i < q$, all but less than $(2d)^{-d}\binom{x}{d}$ d-sets $U \subset V_{i+1}$ satisfy $|N(U) \cap V_i| \geq x$. Then, for every q-partite graph H with n vertices and maximum degree at most $\Delta(H) \leq d$, there are at least $\left(\frac{x}{4}\right)^n$ labeled copies of H in G.*

We will also need the following Turán-type result, from which the currently best known upper bound on the Ramsey number of a bounded degree bipartite graph follows.

Theorem 5 (*Theorem 1.1 from [9]*)**.** *Let H be a bipartite graph with n vertices and maximum degree $\Delta \geq 1$. If $\varepsilon > 0$ and G is a graph with $N \geq 32\Delta\varepsilon^{-\Delta} n$ vertices and at least $\varepsilon\binom{N}{2}$ edges, then H is a subgraph of G.*

Finally we need one simple observation, whose proof we provide here for completeness.

Proposition 6. *Let $F = (V, E)$ be a graph with m edges. Then there exists a subset $U \subseteq V$ with $|U| < \frac{m}{d}$ such that $\Delta(F \setminus U) \leq d$.*

Proof. Let v_1 be a vertex of maximum degree in $F_1 := F$ and set $d_1 := \Delta(F)$. We delete v_1 from F denoting the new graph by F_2. We proceed inductively, deleting from F_i a vertex of maximum degree v_i, setting $d_i := \Delta(F_i)$ and defining the new graph $F_{i+1} := F_i - v_i$ and stop with $F_{|V(F)|+1} = \emptyset$. Let j be the smallest integer with $\Delta(F_{j+1}) \leq d$. Thus, till we obtained F_{j+1} we must have deleted j vertices, each of degree larger than d. Moreover, by the construction of the sequence of v_is, we have $m = |E(F)| = \sum_{i=1}^{|V(F)|} \Delta(F_i)$. Therefore, $jd < m$ and the claim follows with $U := \{v_1, \ldots, v_j\}$. □

Often we try to avoid using floor and ceiling signs as they will not affect our calculations.

3. Multicolor Ramsey number of bipartite graphs with *m* edges

The idea of the proof of Theorem 2 is quite simple. Given a coloring of $E(K_N)$, we will perform a color focussing argument by considering the densest color class and taking a vertex with maximum degree in it. Then we iterate on the colored neighborhood of that vertex. After less than km/d steps we arrive at the situation, where we can embed all m/d vertices from U (of high degree in F) onto the vertices specified in the focussing process, the remaining graph $F - U$ has maximum degree at most d (by Proposition 6) and is bipartite, and thus can be embedded in one round, by Theorem 5.

Proof of Theorem 2. Given a bipartite graph F with m edges and no isolated vertices. We choose with foresight $d = \sqrt{km}$. By Proposition 6, let U be a set of $t = \lfloor m/d \rfloor = \lfloor \sqrt{m/k} \rfloor$ vertices such that $\Delta(F \setminus U) \leq d$. Further observe that $|V(H)| \leq 2m$.