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On automorphisms and structural properties of double generalized Petersen graphs

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ABSTRACT

The concept of double generalized Petersen graphs was introduced by Zhou and Feng in Zhou and Feng (2012), where it was asked for a characterization of the automorphism groups of these graphs. This paper gives this characterization and considers hamiltonicity, vertex-coloring and edge-coloring properties of these graphs.

1. Introduction

The generalized Petersen graphs $\mathrm{GP}(n, k)$, first introduced by Coxeter in [3], are a natural generalization of the well-known Petersen graph (see Fig. 1).

Definition 1.1. Given an integer $n \geq 3$ and $k \in \mathbb{Z}_n \setminus \{0\}$, $2 \leq 2k < n$, the generalized Petersen graph $\mathrm{GP}(n, k)$ is defined to have vertex set $\{u_i, v_i | \, i \in \mathbb{Z}_n\}$ and edge set the union $\Omega \cup \Sigma \cup I$, where

$\Omega = \{\{u_i, u_{i+1}\}, \mid i \in \mathbb{Z}_n\}$ (the outer edges),
$\Sigma = \{\{u_i, v_i\}, \mid i \in \mathbb{Z}_n\}$ (the spokes), and
$I = \{\{v_i, v_{i+k}\}, \mid i \in \mathbb{Z}_n\}$ (the inner edges).

A natural generalization of the generalized Petersen graphs are the double generalized Petersen graphs $\mathrm{DP}(n, t)$, first introduced in [7] as examples of vertex-transitive non-Cayley graphs. They are defined as follows (two examples are given in Fig. 2).

Definition 1.2. Given an integer $n \geq 3$ and $t \in \mathbb{Z}_n \setminus \{0\}$, $2 \leq 2t < n$, the double generalized Petersen graph $\mathrm{DP}(n, t)$ is defined to have vertex set $\{x_i, y_i, u_i, v_i | \, i \in \mathbb{Z}_n\}$ and edge set the union $\Omega \cup \Sigma \cup I$, where

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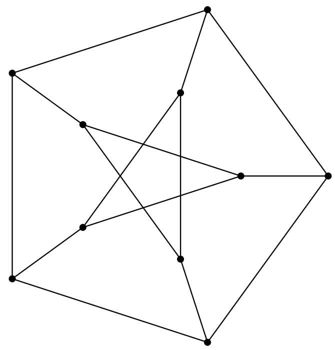

Fig. 1. The generalized Petersen graph GP(5, 2) (the Petersen graph).

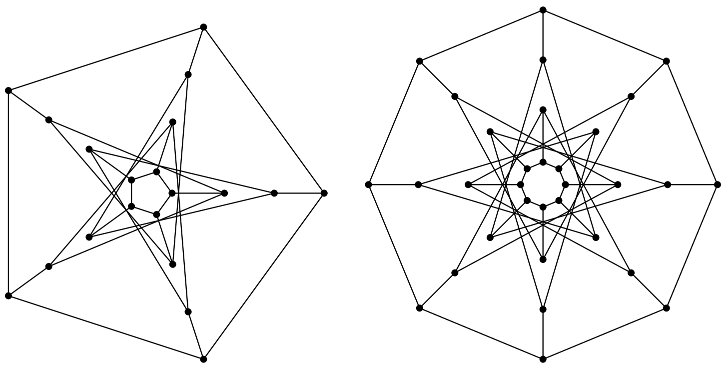

Fig. 2. The double generalized Petersen graph DP(5, 2), which is isomorphic to the generalized Petersen graph GP(10, 2) (the dodecahedron), on the left hand-side picture, and the double generalized Petersen graph DP(8, 3) on the right hand-side picture.

$\Omega = \{\{x_i, x_{i+1}\}, \{y_i, y_{i+1}\} \mid i \in \mathbb{Z}_n\}$ (the outer edges),
$\Sigma = \{\{x_i, u_i\}, \{y_i, v_i\} \mid i \in \mathbb{Z}_n\}$ (the spokes), and
$I = \{\{u_i, v_{i+t}\}, \{v_i, u_{i+t}\} \mid i \in \mathbb{Z}_n\}$ (the inner edges).

The motivation for the research in this paper, resulting in a complete characterization of automorphism groups of double generalized Petersen graphs (see Propositions 3.4 and 3.5, and Remark 3.10), comes from questions post in [7]. The characterization is obtained with a generalization of the method that was used in [5] to obtain a characterization of automorphisms of generalized Petersen graphs.

Aiming at obtaining the information how structural properties of double generalized Petersen graphs are linked with the structural properties of generalized Petersen graphs [1,2], hamiltonicity properties, vertex-coloring and edge-coloring of double generalized Petersen graphs are also considered. In particular, it is shown that any DP$(2n, t)$ has a Hamilton cycles (see Proposition 4.1) whereas for DP$(2n+1, t)$ the existence of a Hamilton cycles is proven only for t being a generator of $\mathbb{Z}_{2n+1}$ (see Proposition 4.2). Any DP$(2n, t)$ is bipartite, thus two colors suffice for proper vertex-coloring whereas for DP$(2n+1, t)$ three colors are needed (see Lemmas 5.1 and 5.2). Finally, it is shown that there are no snarks amongst double generalized Petersen graphs (see Lemma 5.3).

2. Preliminaries

Given a graph X the set of its vertices is denoted by $V(X)$ and the set of its edges by $E(X)$. For a graph X the bijection $\phi : V(X) \to V(X)$ such that $xy \in E(X) \Leftrightarrow \phi(x)\phi(y) \in E(X)$ is called an automorphism of X. The set of all automorphisms of X together with composition of mappings forms a group called the *automorphism group* Aut(X) of X. A graph X is said to be *vertex-transitive* and *edge-transitive* if its automorphism group Aut(X) acts transitively on $V(X)$ and $E(X)$, respectively. The set of neighbors of the vertex v in X is denoted by $N(v) = \{u \in V(X) \mid uv \in E(X)\}$. A Hamilton cycle in a graph is a cycle containing all the vertices of the graph.

3. Automorphisms of double generalized Petersen graphs

In this section, with a generalization of the methods used in [5] to characterize automorphisms of generalized Petersen graphs, we give a complete characterization of automorphism groups of double generalized Petersen graphs, implying