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Arc-transitive graphs of square-free order and small valency

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ABSTRACT

This paper is one of a series of papers devoted to characterizing edge-transitive graphs of square-free order. It presents a complete list of locally-primitive arc-transitive graphs of square-free order and valency $d \in \{5, 6, 7\}$.

1. Introduction

All graphs and groups considered in this paper are assumed to be finite.

Let $\Gamma = (V, E)$ be a simple connected graph with vertex set V and edge set E. The number of vertices $|V|$ is called the *order* of Γ. Let $\mathrm{Aut}\Gamma$ be the automorphism group of Γ and let G be a subgroup of $\mathrm{Aut}\Gamma$, written as $G \le \mathrm{Aut}\Gamma$. Then the graph Γ is said to be *G-vertex-transitive* or *G-edge-transitive* if G acts transitively on V and E, respectively. Recall that an *arc* in Γ is an ordered pair of adjacent vertices. The graph Γ is said to be *G-arc-transitive* if G acts transitively on the set of all arcs in Γ. For $\alpha \in V$, we denote by G_α and $\Gamma(\alpha)$ respectively the stabilizer of α in G and the set of neighbors of α in Γ, that is,

$$G_\alpha = \{g \in G \mid \alpha^g = \alpha\} \quad \text{and} \quad \Gamma(\alpha) = \{\beta \in V \mid \{\alpha, \beta\} \in E\}.$$

The graph Γ is called *G-locally-primitive* if for every $\alpha \in V$ the stabilizer G_α acts primitively on $\Gamma(\alpha)$. It is easy to see that Γ is G-edge-transitive if it is G-locally-primitive. Moreover, if Γ is both G-vertex-transitive and G-locally-primitive, then Γ must be G-arc-transitive; in this case, Γ is said to be *G-locally-primitive arc-transitive*.

The study of graphs with square-free order has a long history, see for example [1,16,17,19] for those graphs of order being a product of two primes. This paper is devoted to classifying arc-transitive graphs of square-free order and small valency.

In recent work [14], the authors gave a reduction for connected locally-primitive arc-transitive of square-free order. We proved that, for a connected locally-primitive arc-transitive graph Γ of square-free order and valency d, if it is not a complete bipartite graph then either $\mathrm{Aut}\Gamma$ is soluble, or Γ is a cover of one of the 'basic' graphs associated with $\mathrm{PSL}(2, p)$, $\mathrm{PGL}(2, p)$ and a finite number (depending only on the valency d) of other almost simple groups. Then for some small values of d we may determine most 'basic' graphs, which makes it possible to give a classification of such graphs of small valencies.

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Thus a natural question is to find a classification of locally-primitive arc-transitive graphs of square-free order and small valency d. This question was solved for $d = 3$ and 4 in [13] and [15], respectively. In this paper we deal with the case where $d \in \{5, 6, 7\}$. Our main result is stated as follows.

Theorem 1.1. *Let Γ be a connected G-locally-primitive arc-transitive graph of square-free order and valency $d = 5, 6$ or 7. Then one of the following statements holds.*

(i) $G = \mathrm{D}_{2n}{:}\mathbb{Z}_d$ *with* $d \in \{5, 7\}$, *and* Γ *is a graph given by* Construction 4.1.
(ii) Γ *is isomorphic to one of the following graphs:*
K_6, K_7, $\mathrm{K}_{5,5}$, $\mathrm{K}_{7,7}$ *and* $\mathrm{K}_{7,7} - 7\mathrm{K}_2$;
the incidence graphs of $\mathrm{PG}(3, 2)$, $\mathrm{PG}(2, 4)$, $\mathrm{PG}(2, 5)$ *and* $\mathrm{GQ}(4)$;
the graphs given in Examples 4.2–4.5.
(iii) $G = \mathrm{PSL}(2, p)$ *or* $\mathrm{PGL}(2, p)$ *for odd prime* p, *and for an edge* $\{\alpha, \beta\}$ *of* Γ *the pair* $(G_\alpha, G_{\alpha\beta})$ *is listed in* Table 3.

For groups, we follow the notation used in the Atlas [6] while we sometimes use $\mathbb{Z}_l$ and $\mathbb{Z}_p^k$ to denote respectively the cyclic group of order l and the elementary abelian group of order p^k.

2. Preliminaries

Let $\Gamma = (V, E)$ be a graph of valency d, let $\{\alpha, \beta\} \in E$ and $G \le \mathrm{Aut}\Gamma$. Set $G_{\alpha\beta} = G_\alpha \cap G_\beta$, call the *arc-stabilizer* of (α, β) (and (β, α)). Assume that Γ is G-arc-transitive. Then G_α is transitive on $\Gamma(\alpha)$, and $d = |\Gamma(\alpha)| = |G_\alpha : G_{\alpha\beta}|$. Take $x \in G$ with $(\alpha, \beta)^x = (\beta, \alpha)$. Then

$$x \in \mathbf{N}_G(G_{\alpha\beta}) \setminus G_{\alpha\beta}, \quad x^2 \in G_{\alpha\beta}.$$

(In particular, the index $|\mathbf{N}_G(G_{\alpha\beta}) : G_{\alpha\beta}|$ is even.) Obviously, this x may be chosen as a 2-element in the normalizer $\mathbf{N}_G(G_{\alpha\beta})$. Moreover, Γ is connected if and only if $\langle x, G_\alpha\rangle = G$. Since G is transitive on V, the map $\alpha^g \mapsto G_\alpha g$ is a bijection between V and $[G : G_\alpha]$, the set of right cosets of G_α in G. It is easy to show that this map is an isomorphism from the graph Γ to a coset graph defined as follows.

Let G be a finite group and H be a core-free subgroup of G, where core-free means that $\cap_{g\in G} H^g = 1$. For $x \in G \setminus H$, the *coset graph* $\mathrm{Cos}(G, H, H\{x, x^{-1}\}H)$ is defined on $[G : H]$ such that Hg_1 and Hg_2 are adjacent whenever $g_2g_1^{-1} \in HxH \cup Hx^{-1}H$. Note that G may be viewed as a subgroup of $\mathrm{AutCos}(G, H, H\{x, x^{-1}\}H)$, where G acts on $[G : H]$ by right multiplication. The following statements for coset graphs are well-known.

Lemma 2.1. *Let G be a finite group and H a core-free subgroup of G. Set $\Gamma = \mathrm{Cos}(G, H, H\{x, x^{-1}\}H)$, where $x \in G \setminus H$. Then Γ is both G-vertex-transitive and G-edge-transitive, and*

(i) *Γ is G-arc-transitive if and only if $HxH = HyH$ for some 2-element $y \in \mathbf{N}_G(H \cap H^x) \setminus H$ with $y^2 \in H \cap H^x$; in this case, Γ has valency $|H : (H \cap H^y)|$;*
(ii) *Γ is connected if and only if $\langle H, x\rangle = G$.*

Let $\Gamma = (V, E)$ be a connected graph and $G \le \mathrm{Aut}\Gamma$. For $\alpha \in V$, the stabilizer G_α induces a permutation group $G_\alpha^{\Gamma(\alpha)}$. Let $G_\alpha^{[1]}$ be the kernel of this action. Then $G_\alpha^{\Gamma(\alpha)} \cong G_\alpha/G_\alpha^{[1]}$. Consider the actions of Sylow subgroups of $G_\alpha^{[1]}$ on V. It is easily shown that the next lemma holds, see [5] for example.

Lemma 2.2. *Let $\Gamma = (V, E)$ be a connected regular graph, $G \le \mathrm{Aut}\Gamma$ and $\alpha \in V$. Assume that $G_\alpha \ne 1$. Let p be a prime divisor of $|G_\alpha|$. Then $p \le |\Gamma(\alpha)|$. If further Γ is G-vertex-transitive, then p divides $|G_\alpha^{\Gamma(\alpha)}|$ and, for $\beta \in \Gamma(\alpha)$, each prime divisor of $|G_{\alpha\beta}|$ is less than $|\Gamma(\alpha)|$.*

Lemma 2.3. *Assume that $\Gamma = (V, E)$ is a connected G-vertex-transitive graph. Let $N \lhd G$ be a normal subgroup of G such that $N_\alpha^{\Gamma(\alpha)}$ is semiregular for some $\alpha \in V$. Then $N_\alpha^{[1]} = 1$, that is, N_α is faithful on $\Gamma(\alpha)$.*

Proof. Let $\beta \in \Gamma(\alpha)$. Then $\beta = \alpha^x$ for some $x \in G$, and hence $N_\beta = N \cap G_{\alpha^x} = (N_\alpha)^x$. It follows that $N_\beta^{\Gamma(\beta)}$ and $N_\alpha^{\Gamma(\alpha)}$ are permutation isomorphic; in particular, $N_\beta^{\Gamma(\beta)}$ is semiregular on $\Gamma(\beta)$. Thus $N_\alpha^{[1]}$ acts trivially on $\Gamma(\beta)$, and so $N_\alpha^{[1]} = N_\beta^{[1]}$. Since Γ is connected, $N_\alpha^{[1]}$ fixes each vertex of Γ, and hence $N_\alpha^{[1]} = 1$. □

Lemma 2.4. *Let $\Gamma = (V, E)$ be a connected graph, $N \lhd G \le \mathrm{Aut}\Gamma$ and $\alpha \in V$. Assume that either N is regular on V, or Γ is a bipartite graph such that N is regular on both the bipartition subsets of Γ. Then $G_\alpha^{[1]} = 1$.*

Proof. Set $X = NG_\alpha^{[1]}$. Then $X_\alpha = G_\alpha^{[1]}$ and $X_\alpha^{[1]} = G_\alpha^{[1]}$, and hence $X_\alpha^{\Gamma(\alpha)} = 1$.

Assume first that N is regular on V. Then $G = NG_\alpha$. It follows that X is normal in G. Thus our result follows from Lemma 2.3.

Now assume that Γ is a bipartite graph with bipartition subsets U and W, and that N is regular on both U and W. For each $\delta \in U \cup W$, we have $NX_\alpha = X = NX_\delta$, and $|X_\delta| = |X_\alpha|$. Since $X_\alpha = G_\alpha^{[1]}$ acts trivially on $\Gamma(\alpha)$, we have $X_\alpha \le X_\beta$ for