Discrete Mathematics 339 (2016) 2941–2954

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

# A coloring problem for intersecting vector spaces

Carlos Hoppen [a,*], Hanno Lefmann [b], Knut Odermann [b]

[a] *Instituto de Matemática, UFRGS – Avenida Bento Gonçalves, 9500, 91501–970 Porto Alegre, RS, Brazil*

[b] *Fakultät für Informatik, Technische Universität Chemnitz, Straße der Nationen 62, D-09107 Chemnitz, Germany*

ARTICLE INFO

*Article history:*
Received 1 October 2014
Received in revised form 2 June 2016
Accepted 3 June 2016
Available online 1 July 2016

*Keywords:*
Finite vector spaces
Erdős–Ko–Rado Theorem
Coloring
Stability

ABSTRACT

In Hoppen et al. (2012) Kohayakawa and two of the current authors considered a variant of the classical Erdős–Ko–Rado problem for families of $\ell$-intersecting $r$-sets in which they asked for the maximum number of edge-colorings of an $n$-vertex $r$-uniform hypergraph such that all color classes are $\ell$-intersecting. This resulted in a fairly complete characterization of the corresponding extremal families. In this paper, we show that, when the number of colors is $k \in \{2, 3, 4\}$, similar results may be obtained in the context of vector spaces. In particular, we observe that a rather unusual instability phenomenon occurs for $k = 4$ colors, namely that the problem is unstable despite admitting a unique extremal configuration up to isomorphism.



## 1. Introduction and main results

We consider a $q$-analogue of a coloring problem based on the Erdős–Ko–Rado Theorem [6] for families of $\ell$-intersecting $r$-sets, which determines the size of a largest family of $r$-subsets of an $n$-element set with the property that any two sets have nonempty intersection (or, more generally, have intersection of cardinality at least $\ell$). Kohayakawa and two of the current authors [11] proposed the following twist to this problem: instead of looking for the largest $\ell$-intersecting family of $r$-sets in an $n$-element set, they were interested in *colorings* of families of $r$-sets such that *every color class is* $\ell$*-intersecting*. For a fixed number $k \geq 2$ of colors and for positive integers $n$, $r$ and $\ell$, they asked for $\ell$-intersecting families of $r$-subsets of an $n$-element set that admit the largest number of $k$-colorings of this type. Such coloring questions had already been considered in the context of hypergraphs [15,16,14], but previous results were typically restricted to the case $k \in \{2, 3\}$ and consisted of instances where the extremal configurations of the classical problem would still be optimal for its coloring counterpart. The work in [11] was significant in two respects. On the one hand, the authors managed to describe the unique extremal configuration for a diverse set of parameters, including several cases where it does not coincide with the extremal configuration of the uncolored case. On the other hand, they found parameters for which the problem is provably unstable despite admitting a unique extremal configuration up to isomorphism (see also [12]).

In this paper, we address a coloring version of the Erdős–Ko–Rado Theorem for Vector Spaces [8]. To the best of our knowledge, this is the first time that a coloring problem of this type has been investigated for finite vector spaces. There has been considerable interest in extending results about $r$-subsets of an $n$-element set to results about $r$-dimensional subspaces of a finite $n$-dimensional vector space. For instance, Frankl and Wilson [8] determined the maximum number of elements in an $\ell$-intersecting family of linear $r$-dimensional subspaces of an $n$-dimensional vector space $V_n$, that is, in a family of $r$-dimensional subspaces for which the intersection of any pair of spaces has dimension at least $\ell$. They showed that, for

* Corresponding author.
*E-mail addresses:* choppen@ufrgs.br (C. Hoppen), Lefmann@Informatik.TU-Chemnitz.de (H. Lefmann), knut.odermann@Informatik.TU-Chemnitz.de (K. Odermann).

http://dx.doi.org/10.1016/j.disc.2016.06.007

sufficiently large $n$, this maximum is achieved by the family of all $r$-dimensional subspaces of $V_n$ that contain some fixed $\ell$-dimensional subspace, which is a natural translation of the Erdős–Ko–Rado Theorem to vector spaces, and therefore carries this name. More recently, Blokhuis et al. [2] obtained a vector-space analogue of the classical Hilton–Milner Theorem [10] for set systems, which states that 1-intersecting families such that no element lies in all of its sets are substantially smaller than the largest intersecting family (see Frankl [7] for an extension to $\ell$-intersecting families). For more results relating extremal set theory with finite geometry, we refer the reader to Blokhuis, Brouwer, Szőnyi and Weiner [3].

Our results naturally extend set-theoretical work in [11,12] to the realm of finite vector spaces provided that the number of colors is $k \le 4$. As expected, the additional algebraic structure of vector spaces creates technical difficulties that are absent in the set situation. For instance, this prevented us from adapting the general proof strategy of [11,12] to the case $k \ge 5$ (see Section 5 for further discussion about obstructions in this case). However, we were able to show that the two main features of the solution for set systems also hold for vector spaces. For $k \in \{2, 3\}$ colors and sufficiently large $n$, the unique extremal configuration is given by the extremal configuration of the Erdős–Ko–Rado Theorem for Vector Spaces, with the exception of the case $n = 2r$, in which there are two extremal configurations.

For $k = 4$ colors and large dimension $n$ the extremal configuration is not given by the extremal configuration of the Erdős–Ko–Rado Theorem for Vector Spaces; moreover, the problem is provably unstable in this case, despite admitting a unique extremal configuration up to isomorphism. More precisely, given positive integers $n$, $r$, $k$ and $\ell$, and a prime power $q$, let $\mathcal{P}_{n,q,r,k,\ell}$ be the problem of maximizing, over all families of $r$-dimensional subspaces of an $n$-dimensional vector space over a finite field $GF(q)$, the number of $k$-colorings such that any two subspaces with the same color share an $\ell$-dimensional subspace. This problem may be analyzed in the context of 'stability' as in the Simonovits Stability Theorem [18] for graphs. Roughly speaking, the problem of maximizing a function $f$ over a class $\mathcal{A}$ of combinatorial objects is said to be *stable* if every object that is very close to maximizing $f$ is almost equal to an isomorphic copy of the unique object that maximizes $f$. For instance, the Hilton–Milner Theorem implies that the problem of finding a largest intersecting family of $r$-subsets of some $n$-sets is stable, as it shows that any intersecting family that does not have the structure given by the Erdős–Ko–Rado Theorem is substantially smaller than an optimal family. In this paper, we prove that, provided $n > 2r$, the problem $\mathcal{P}_{n,q,r,k,\ell}$ is stable for $k \in \{2, 3\}$ and for $k = 4$ and $\ell = 1$, but it is unstable for $k = 4$ and $\ell \ge 2$. The latter provides an instance of a natural problem that is provably unstable, but which has a unique optimal solution up to isomorphism. We are aware of only very few problems that share this property. Other than [12], which was already mentioned for uniform hypergraphs, we may cite [4] in the context of multigraphs.

Before proceeding, we should mention that Erdős and Rothschild [5] were the first to consider a coloring problem of this type, which was a variant of the classical Turán problem [20]. For progress in this direction, we refer the reader to Pikhurko and Yilma [17], and the references therein.

We start with some notation. All vector spaces in this paper are *linear*. Consider an $n$-dimensional vector space $V_n$ over $GF(q)$, where as usual $GF(q)$ denotes the finite field with $q$ elements. Let $\dim(U)$ be the dimension of a vector space $U$. For short, we call an $r$-dimensional vector space an *$r$-space* if there is no ambiguity. Also, for a vector space $U$ over $GF(q)$, the set of all $r$-spaces in $U$ is denoted by $\binom{U}{r}_q$. Given a positive integer $r \le n$, the number of $r$-spaces in $V_n$ over $GF(q)$ is given by the *Gaussian coefficient*

$$\binom{n}{r}_q = \frac{(q^n - 1)(q^{n-1} - 1)\cdots(q^{n-r+1} - 1)}{(q^r - 1)(q^{r-1} - 1)\cdots(q - 1)}.$$

Let $\mathcal{F}$ be a family of $r$-spaces in $V_n$. For a fixed integer $\ell$, where $0 < \ell < r$, the family $\mathcal{F}$ is called *$\ell$-intersecting* if $\dim(F \cap F') \ge \ell$ for all $F, F' \in \mathcal{F}$.

**Definition 1.1.** Let $\mathcal{F}$ be a family of $r$-spaces in an $n$-space $V_n$ over $GF(q)$. A *$(k, \ell)$-coloring* of $\mathcal{F}$ is a function $\Delta : \mathcal{F} \to [k]$ associating a color with each $r$-space in $\mathcal{F}$ with the property that any two $r$-spaces $F_1, F_2 \in \mathcal{F}$ with the same color are $\ell$-intersecting. If the family $\mathcal{F}$ admits a $(k, \ell)$-coloring, then it is called *$(k, \ell)$-colorable*, and the number of $(k, \ell)$-colorings of $\mathcal{F}$ is denoted by $C_{(k,\ell)}(\mathcal{F})$.

Define $\mathcal{S}_{n,q,r}$ to be the family of all $r$-spaces in an $n$-space $V_n$ over $GF(q)$. Given positive integers $n, q, r, \ell, k$, where $n > r > \ell$ and $q$ is a prime power, we consider the function

$$\chi_{(r,k,\ell,q)}(n) = \max_{\mathcal{F} \subseteq \mathcal{S}_{n,q,r}} \left\{ C_{(k,\ell)}(\mathcal{F}) \right\},$$

that is, $\chi_{(r,k,\ell,q)}(n)$ is the maximum number of $(k, \ell)$-colorings over all families $\mathcal{F}$ of $r$-spaces in an $n$-space over $GF(q)$. A family $\mathcal{F}$ for which $\chi_{(r,k,\ell,q)}(n) = C_{(k,\ell)}(\mathcal{F})$ is called *$(k, \ell)$-extremal*.

The problem of determining $\chi_{(r,k,\ell,q)}(n)$ and the corresponding $(k, \ell)$-extremal families of $r$-spaces in an $n$-space $V_n$ over $GF(q)$ turns out to be related with the Erdős–Ko–Rado Theorem for vector spaces. This theorem is due to Frankl and Wilson [8], and was later complemented by Tanaka [19], who showed that, in the case $n = 2r$, there are exactly two non-isomorphic maximum families. We state the theorem only in the case $n > 2r - \ell$, as otherwise every pair of $r$-spaces in $V_n$ is $\ell$-intersecting, so that $C_{(k,\ell)} = k^{|\mathcal{F}|}$ for every $\mathcal{F} \subseteq \mathcal{S}_{n,q,r}$. Hence $\chi_{(r,k,\ell,q)}(n) = k^{\binom{n}{r}_q}$, with equality if and only if $\mathcal{F} = \mathcal{S}_{n,q,r}$.