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A non-classical unital of order four with many translations

Theo Grundhöfer [a], Markus J. Stroppel [b], Hendrik Van Maldeghem [c,*]

[a] Institut für Mathematik, Universität Würzburg, Am Hubland, D-97074 Würzburg, Germany
[b] LExMath, Fakultät für Mathematik und Physik, Universität Stuttgart, D-70550 Stuttgart, Germany
[c] Department of Mathematics, Ghent University, Krijgslaan 281, S23, B-9000 Gent, Belgium

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ABSTRACT

We give a general construction for unitals of order q admitting an action of $\mathrm{SU}(2, q)$. The construction covers the classical Hermitian unitals, Grüning's unitals in Hall planes and at least one unital of order four where the translation centers fill precisely one block. For the latter unital, we determine the full group of automorphisms and show that there are no group-preserving embeddings into (dual) translation planes of order 16.

Introduction

Let $\mathbb{U} = (U, \mathcal{B})$ be a unital of order q, and let $\Gamma := \mathrm{Aut}(\mathbb{U})$. For each point $c \in U$ we consider the group $\Gamma_{[c]}$ of translations with center c, i.e., the set of all automorphisms of $\mathbb{U}$ fixing each block through c. We say that c is a translation center of $\mathbb{U}$ if $\Gamma_{[c]}$ is transitive on the set of points different from c on any block through c.

The main result of [9] states that the unital $\mathbb{U}$ is classical (i.e., isomorphic to the Hermitian unital corresponding to the field extension $\mathbb{F}_{q^2}/\mathbb{F}_q$) if it has non-collinear translation centers. Unitals with precisely one translation center seem to exist in abundance (we indicate several quite different classes of examples in Section 5 below). If there are two translation centers c and c' then the orbit of c' under $\Gamma_{[c]}$ fills the complement of c in the block joining c with c'. We give an example of a unital (of order 4, see Section 1 below) where the translation centers fill just one block. As far as we know, this unital is the first (and up to now, the only) one with that property.

1. A curious unital of order four

Let $\mathrm{Syl}_2(\mathrm{A}_5)$ be the set of all five Sylow 2-subgroups in the alternating group A_5, and let $S := \langle(01234)\rangle$. We consider the subsets $E_1 := \{\mathrm{id}, (023), (024), (123), (03421)\}$ and $E_2 := E_1^{(1243)} = \{\mathrm{id}, (041), (043), (124), (01342)\}$. For later reference, we abbreviate $\mathcal{E} := \{E_1, E_2\}$.

We construct an incidence structure $\mathbb{U}_{\mathcal{E}}$ with two kinds of points: elements of A_5 and elements of $\mathrm{Syl}_2(\mathrm{A}_5)$. The blocks are the following:

* Corresponding author.
E-mail addresses: grundhoefer@mathematik.uni-wuerzburg.de (T. Grundhöfer), stroppel@mathematik.uni-stuttgart.de (M.J. Stroppel), hvm@cage.UGent.be (H. Van Maldeghem).

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- cosets Tg for $T \in \mathrm{Syl}_2(\mathrm{A}_5)$, $g \in \mathrm{A}_5$,
- cosets Sg for $g \in \mathrm{A}_5$,
- sets $E_j g$ for $g \in \mathrm{A}_5$ and $j \in \{1, 2\}$,
- a single block named $[\infty]$.

Note that the elements of $\mathrm{Syl}_2(\mathrm{A}_5)$ are used as (labels for) points and also as certain blocks. Incidence between points in A_5 and subsets is the obvious one. A point $T \in \mathrm{Syl}_2(\mathrm{A}_5)$ is incident with $[\infty]$, with each *left* coset gT $(= T^g g)$, and with no other block.

In the present paper, we show that the incidence geometry $\mathbb{U}_{\mathcal{E}}$ is a non-classical unital of order 4, with the following properties: The action of A_5 by multiplication from the right on itself and on the blocks apart from $[\infty]$ is an action by automorphisms of $\mathbb{U}_{\mathcal{E}}$. Each $T \in \mathrm{Syl}_2(\mathrm{A}_5)$ acts by translations with center T. There are no other translations of $\mathbb{U}_{\mathcal{E}}$. The full automorphism group of $\mathbb{U}_{\mathcal{E}}$ is isomorphic to a semi-direct product of A_5 with a cyclic group of order four, where a generator of that cyclic group induces conjugation by (1243) on A_5.

2. A general construction

Motivated by the action (see Section 3.1 below) of $\mathrm{SU}(2, q)$ on the classical (Hermitian) unital of order q, we study geometries as follows.

Lemma 2.1. *Let G be a group, let T be a subgroup such that conjugates T^g and T^h have trivial intersection unless they coincide (i.e., the conjugacy class T^G forms a T.I. set). Assume that there is a subgroup S and a collection $\mathcal{D}$ of subsets of G such that each set $D \in \mathcal{D}$ contains 1, and the following properties hold:*

(Q) *For each $D \in \mathcal{D}$, the map $(D \times D) \setminus \{(x, x) \mid x \in D\} \to G\colon (x, y) \mapsto xy^{-1}$ is injective.*
We abbreviate $D^* := \{xy^{-1} \mid x, y \in D, x \neq y\}$.

(P) *The system consisting of $S \setminus \{1\}$, all conjugates of $T \setminus \{1\}$ and all sets D^* with $D \in \mathcal{D}$ forms a partition of $G \setminus \{1\}$.*

Then the incidence structure with point set G and block set

$$\mathcal{B}^{\infty} := \{Sg \mid g \in G\} \cup \{T^h g \mid h, g \in G\} \cup \{Dg \mid D \in \mathcal{D}, g \in G\}$$

is a linear space. Each involution of G is contained in $S \cup \bigcup_{g \in G} T^g$.

We consider each conjugate T^h as a point at infinity, call $[\infty] := \{T^h \mid h \in G\}$ the block at infinity (incident with each point at infinity, and no point in G), and extend the incidence relation in two different ways:

(a) *Make each conjugate T^h incident with each coset $T^{hg^{-1}}g = gT^h$ (and no other block in $\mathcal{B}^{\infty}$). This gives an incidence structure $\mathbb{U}_{\mathcal{D}} := \left(G \cup \{T^h \mid h \in G\}, \mathcal{B}^{\infty} \cup \{[\infty]\}, I\right)$.*

(b) *Make each conjugate T^h incident with each coset $T^h g$ (and no other block in $\mathcal{B}^{\infty}$). This gives an incidence structure $\mathbb{U}^{\flat}_{\mathcal{D}} := \left(G \cup \{T^h \mid h \in G\}, \mathcal{B}^{\infty} \cup \{[\infty]\}, I^{\flat}\right)$.*

Then both $\mathbb{U}_{\mathcal{D}}$ and $\mathbb{U}^{\flat}_{\mathcal{D}}$ are linear spaces, and the following hold.

1. *Via multiplication from the right on G and conjugation on the point row of $[\infty]$, the group G acts as a group of automorphisms on $\mathbb{U}_{\mathcal{D}}$.*
2. *On $\mathbb{U}^{\flat}_{\mathcal{D}}$ the group G also acts by automorphisms via multiplication from the right on G but trivially on the point row of $[\infty]$.*

Now let G be finite, and abbreviate $q := |T|$. Assume that $|G| = q^3 - q$, that there are $q + 1$ conjugates of T, and that $|S| = q + 1 = |D|$ holds for each $D \in \mathcal{D}$. Note that we have $|\mathcal{D}| = q - 2$ in that case.

3. *Both $\mathbb{U}_{\mathcal{D}}$ and $\mathbb{U}^{\flat}_{\mathcal{D}}$ are $2 - (q^3 + 1, q + 1, 1)$ designs; i.e., unitals of order q.*
4. *On the unital $\mathbb{U}_{\mathcal{D}}$ each conjugate of T acts as a group of translations. Thus each point on the block $[\infty]$ is a translation center, and G is two-transitive on $[\infty]$.*
5. *On the unital $\mathbb{U}^{\flat}_{\mathcal{D}}$ the group G contains no translation except the trivial one.*

Proof. Assume first that some involution $s \in G$ lies outside $S \cup \bigcup_{g \in G} T^g$. Then the assumptions yield that s is of the form $s = xy^{-1}$ with $x, y \in D$ for some $D \in \mathcal{D}$. Then $xy^{-1} = s = s^{-1} = yx^{-1}$ contradicts assumption **(Q)**.

We consider blocks through 1 first. Cosets like $T^h g$ or Sg contain 1 if, and only if, they coincide with the subgroups T^h and S, respectively. The situation is different for Dg with $D \in \mathcal{D}$: here $1 \in Dg \iff g^{-1} \in D$. Thus Dg passes through 1 precisely if $Dg \subseteq D^* \cup \{1\}$. Now the partition required in condition **(P)** secures that each element in $G \setminus \{1\}$ is joined to 1 by a unique block in $\mathcal{B}^{\infty}$. As G forms a transitive group of automorphisms of $(G, \mathcal{B}^{\infty})$, that structure is a linear space.

The conditions imposed on the orders of G, T, S, and $|D|$ make it immediate that both $\mathbb{U}_{\mathcal{D}}$ and $\mathbb{U}^{\flat}_{\mathcal{D}}$ are $2-(q^3+1, q+1, 1)$ designs.

Each orbit of each conjugate of T is contained in a block, but these blocks are assigned points at infinity in different ways in the two incidence structures $\mathbb{U}_{\mathcal{D}}$ and $\mathbb{U}^{\flat}_{\mathcal{D}}$. For $(g, T^h) \in G \times [\infty]$ the unique joining block in $\mathbb{U}_{\mathcal{D}}$ is $T^{hg^{-1}}g$. Thus $\mathbb{U}_{\mathcal{D}}$ is a linear space. In $\mathbb{U}^{\flat}_{\mathcal{D}}$ the unique joining block for (g, T^h) is $T^h g$, and $\mathbb{U}^{\flat}_{\mathcal{D}}$ is a linear space, as well.