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Note

On a symmetric q-series identity

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ABSTRACT

We prove an interesting symmetric q-series identity which generalizes a result due to Ramanujan. A proof that is analytic in nature is offered, and a bijective-type proof is also outlined.

1. Introduction

In [1] Andrews gives a wonderful introduction of Ramanujan's "Lost" notebook, and lists some interesting identities contained therein. One of which is the following beautiful symmetric identity [1, eq. (1.5)], where if

$$f(\alpha,\beta) := \frac{1}{1-\alpha} + \sum_{n\geq 1} \frac{\beta^n}{(1-\alpha x^n)(1-\alpha x^{n-1}y)(1-\alpha x^{n-2}y^2)\cdots(1-\alpha y^n)}, \tag{1.1}$$

then

$$f(\alpha,\beta) = f(\beta,\alpha). \tag{1.2}$$

The identity we present here is a refinement of the case where $x = q$, and $y = q^2$. Andrews provides an elegant bijective proof of this identity in [1, pg. 107] by taking the conjugate partition (see also Pak [8, pg. 18] for a nice presentation of Andrews' bijection). We will also consider conjugate partitions in the third section, but will require a slightly different approach using a 2-modular diagram (conceptually) to prove the following theorem bijectively. We first note some notation which may be found in [2,6]. We put, throughout this paper, $(a)_n = (a;q)_n := \prod_{0\leq k<n}(1-aq^k)$. Of course the reader should note the infinite product that is obtained by passing the limit $n \to \infty$, which we denote by $(a)_\infty$.

Theorem 1.1. *We have, for arbitrary a, and* $|b| < 1$, $|t| < 1$,

$$\sum_{n\geq 0} \frac{(-abq^{n+1};q)_n t^n}{(bq^n;q)_{n+1}} = \sum_{n\geq 0} \frac{(-atq^{n+1};q)_n b^n}{(tq^n;q)_{n+1}}. \tag{1.3}$$

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2. An analytic proof

Since in [1] an analytic proof uses the q-binomial theorem, we decided to stay with our original use of a different q-polynomial identity. Namely, we use the q-Pfaff–Saalschütz [6, pg. 355, eq. (II.12)]

$$\sum_{n\geq 0}\frac{(a)_n(b)_n(q^{-N})_nq^n}{(c)_n(q)_n(q^{1-N}ab/c)_n}=\frac{(c/a)_N(c/b)_N}{(c)_N(c/ab)_N}. \tag{2.1}$$

The left side of (2.1) may be written

$$\frac{(q)_N}{(c/(ab))_N}\sum_{n\geq 0}\frac{(a)_n(b)_n(c/(ab))_{N-n}}{(c)_n(q)_n(q)_{N-n}}(c/ab)^n. \tag{2.2}$$

Putting $c=bq$ in this identity we obtain

$$\sum_{n\geq 0}\frac{(a)_n(q/a)_{N-n}}{(q)_n(q)_{N-n}(1-bq^n)}(q/a)^n=\frac{(bq/a)_N}{(b)_{N+1}}. \tag{2.3}$$

Now we may use (2.3) to compute the following:

$$\sum_{n\geq 0}\frac{(abq^{n+1};q)_nt^n}{(bq^n;q)_{n+1}}=\sum_{N\geq 0}\sum_{n\geq 0}\frac{(a)_n(q/a)_{N-n}(q/a)^nt^N}{(q)_n(1-bq^{N+n})(q)_{N-n}},$$

shifting summation indices $N\to N+n$ and applying [5, pg. 18, eq. (16.3)] gives,

$$\sum_{N\geq 0}\sum_{n\geq 0}\frac{(a)_n(q/a)_N(q/a)^nt^{N+n}}{(q)_n(1-bq^{N+2n})(q)_N}=\sum_{n\geq 0}\frac{t^n(a)_n(q/a)^n}{(q)_n}\frac{(tq/a)_\infty}{(t)_\infty}\sum_{N\geq 0}\frac{(t)_N}{(tq/a)_N}(bq^{2n})^N.$$

By [5, pg. 4, eq. (6.2)] we compute this to be equal to

$$\frac{(tq/a)_\infty}{(t)_\infty}\sum_{n\geq 0}\frac{(t)_n}{(tq/a)_n}b^n\frac{(tq^{2n+1})_\infty}{(tq^{2n+1}/a)_\infty},$$

which may be simplified to the desired identity,

$$\sum_{n\geq 0}\frac{(atq^{n+1};q)_nb^n}{(tq^n;q)_{n+1}}.$$

3. A bijective proof and some corollaries

We first start with some standard notation on partitions, which can be found in [5, pg. 37]. We write a partition π to be a sequence which consists of nonnegative integers, say $(\pi_1,\pi_2,\ldots,\pi_m)$ where we say each π_i for $1\leq i\leq m$ is a 'part' with the largest π_1, and smallest π_m. The number of such parts is denoted $l(\pi)$, and the number of odd parts will be denoted $o(\pi)$. Since Guo obtained a similar symmetric q-series identity using partitions where odd parts do not repeat, we consider a similar approach. The main bijection appears to be due to R. Chapman in his proof of identities from [3] (see [4] and [6] for more details). We will require an extra step in dealing with the inequality on parts that is in our identity, which is a key difference, however. We may replace q by q^2 in (1.3) and then replace a with aq^{-1} to obtain that

$$\sum_{n\geq 0}\frac{(-abq^{2n+1};q^2)_nt^n}{(bq^{2n};q^2)_{n+1}}=\sum_{n\geq 0}\frac{(-atq^{2n+1};q^2)_nb^n}{(tq^{2n};q^2)_{n+1}}. \tag{3.1}$$

Now, on the left hand side, a keeps track of the number of odd parts, b keeps track of the number of parts and t keeps track of the largest part. It can then be seen that if we let O be the set of partitions where odd parts do not repeat, we have that

$$\sum_{\substack{\pi\in O\\ l(\pi)=j\\ \pi_1\leq 4M\\ \pi_m\geq 2M}}a^{o(\pi)}q^{|\pi|}=\sum_{\substack{\pi\in O\\ l(\pi)=M\\ \pi_1\leq 4j\\ \pi_m\geq 2j}}a^{o(\pi)}q^{|\pi|}, \tag{3.2}$$

which has a similar resemblance (as is to be expected) to Guo's partition identity [7, Theorem 1.2]. The key difference is the inequality on the largest and smallest parts. This indeed causes a problem with using Chapman's bijection directly, but we have a simple solution to this. We say a nonempty partition π is in O^* if its largest part $\pi_1=2M$ is even, and the multiplicity of $2M$ in π is at least the number of smaller parts in π. In our case we start with the left side of (3.2), and if parts are $\geq 2r$, $r\in\mathbb{N}$, say, then $2r$ is removed from each part to appear as a separate part. This process ensures that in the new partition, $2r$