Discrete Mathematics 339 (2016) 2998–3004

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Efron's coins and the Linial arrangement☆

Gábor Hetyei

Department of Mathematics and Statistics, UNC-Charlotte, Charlotte, NC 28223-0001, United States

ARTICLE INFO

Article history:
Received 29 February 2016
Received in revised form 14 June 2016
Accepted 14 June 2016
Available online 18 July 2016

Keywords:
Nontransitive dice
Semiacyclic tournament

ABSTRACT

We characterize the tournaments that are dominance graphs of sets of (unfair) coins in which each coin displays its larger side with greater probability. The class of these tournaments coincides with the class of tournaments whose vertices can be numbered in a way that makes them semiacyclic, as defined by Postnikov and Stanley. We provide an example of a tournament on nine vertices that cannot be made semiacyclic, yet it may be represented as a dominance graph of coins, if we also allow coins that display their smaller side with greater probability. We conclude with an example of a tournament with 81 vertices that is not the dominance graph of any system of coins.

© 2016 Elsevier B.V. All rights reserved.

0. Introduction

A fascinating paradox in probability theory is due to B. Efron, who devised a four element set of nontransitive dice [5]: the first has four faces labeled 4 and two faces labeled 0, the second has all faces labeled 3, the third has four faces labeled 2 and two faces labeled 6, the fourth has three faces labeled 1 and three faces labeled 5. On this list, die number i defeats die number $i+1$ in the cyclic order in the following sense: when we roll the pair of dice simultaneously, die number i is more likely to display the larger number than die number $i+1$. The paradox arose the interest of Warren Buffet, it inspired several other similar constructions and papers in game theory and probability: sample references include [1,3,6,7,12].

This paper intends to investigate a hitherto unexplored aspect of Efron's original example: on each of its dice, only at most two numbers appear. They could be replaced with unfair coins, which display one of two numbers with a given probability. This restriction seems to be strong enough that we should be able to describe exactly which tournaments can be realized as *dominance graphs* of collections of unfair coins, where the direction of the arrows indicates which coin of a given pair is more likely to display the larger number.

Our paper gives a complete characterization in the case when we restrict ourselves to the use of *winner* coins: these are coins that are more likely to display their larger number. The answer, stated in Theorem 3.2, is that a tournament has such a representation exactly when its vertices may be numbered in a way that it becomes a *semiacyclic tournament*. These tournaments were introduced by Postnikov and Stanley [11], the number of semiacyclic tournaments on n numbered vertices is the same as the number of regions of the $(n-1)$-dimensional Linial arrangement. Our result allows to construct an example of a tournament on 9-vertices that cannot be represented using winner coins only. On the other hand, we will see that this example is representable if we also allow the use of *loser* coins, that is, coins that are more likely to display their smaller number. However, as we will see in Theorem 4.3, any tournament that is not representable with a set of winner coins only, gives rise via a direct product operation to a tournament that is not representable by any set of coins. In particular, we obtain an example of a tournament on 81 vertices that cannot be represented by any set of coins as a dominance graph. Our results motivate several open questions, listed in the concluding Section 5.

☆ To (Richard P. S.)^2.

E-mail address: ghetyei@uncc.edu.
URL: http://www.math.uncc.edu/~ghetyei/.

http://dx.doi.org/10.1016/j.disc.2016.06.012
0012-365X/© 2016 Elsevier B.V. All rights reserved.

1. Preliminaries

A hyperplane arrangement is a finite collection of codimension one hyperplanes in a finite dimensional vector space, together with the induced partition of the space into regions. The number of these regions may be expressed in terms of the Möbius function in the intersection poset of the hyperplanes, using Zaslavsky's formula [14].

The *Linial arrangement* $\mathcal{L}_{n-1}$ is the hyperplane arrangement

$$x_i - x_j = 1, \quad 1 \le i < j \le n \tag{1.1}$$

in the $(n-1)$-dimensional vector space $V_{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$. We will use the combinatorial interpretation of the regions of $\mathcal{L}_{n-1}$ in terms of *semiacyclic tournaments*, due to Postnikov and Stanley [11]. A *tournament* on the vertex set $\{1, \ldots, n\}$ is a directed graph with no loops nor multiple edges, such that for each 2-element subset $\{i, j\}$ of $\{1, \ldots, n\}$, exactly one of the directed edges $i \to j$ and $j \to i$ belongs to the graph. We may think of a tournament as the visual representation of the outcomes of all games in a championship, such that each team plays against each other team exactly once, and there is no "draw".

Definition 1.1. A directed edge $i \to j$ is called an *ascent* if $i < j$ and it is a *descent* if $i > j$. For any directed cycle $C = (c_1, \ldots, c_m)$ we denote the number of directed edges that are ascents in the cycle by $\mathrm{asc}(C)$, and the number of directed edges that are descents by $\mathrm{desc}(C)$. A cycle is *ascending* if it satisfies $\mathrm{asc}(C) \ge \mathrm{desc}(C)$. A tournament on $\{1, \ldots, n\}$ is *semiacyclic* if it contains no ascending cycle.

To each region R in $\mathcal{L}_{n-1}$ we may associate a tournament on $\{1, \ldots, n\}$ as follows: for each $i < j$ we set $i \to j$ if $x_i > x_j + 1$ and we set $j \to i$ if $x_i < x_j + 1$. Postnikov and Stanley, and independently Shmulik Ravid, gave the following characterization of the tournaments arising this way [11, Proposition 8.5].

Proposition 1.2. *A tournament T on $\{1, \ldots, n\}$ corresponds to a region R in $\mathcal{L}_{n-1}$ if and only if T is semiacyclic. Hence the number $r(\mathcal{L}_{n-1})$ of regions of $\mathcal{L}_{n-1}$ is the number of semiacyclic tournaments on $\{1, \ldots, n\}$.*

2. The coin model and its elementary properties

In this paper we will study n element sets of (fair and unfair) coins. Each coin is described by a triplet of real parameters (a_i, b_i, x_i) where $a_i \le b_i$ and $x_i > 0$ hold (here $i = 1, 2, \ldots, n$). The ith coin has the number a_i on one side and b_i on the other. After flipping it, it shows the number a_i with probability $1/(1 + x_i)$, equivalently it shows the number b_i with probability $x_i/(1+x_i)$. Note that, as x_i ranges over the set of all positive real numbers, the probability $1/(1+x_i)$ ranges over all numbers in the open interval $(0, 1)$. We call the triplet (a_i, b_i, x_i) the *type* of the coin. We say that coin i *dominates* coin j if, after tossing both at the same time, the probability that coin i displays a strictly larger number than coin j is greater than the probability that coin j displays a strictly larger number. In other words, when we flip both coins, the one displaying the larger number "wins", the other one "loses", and we consider both coins displaying the same number a "draw". The coin that is more likely to win, dominates the other.

We represent the domination relation as the *dominance graph*, whose vertices are the coins and there is a directed edge $i \to j$ exactly when coin i dominates coin j. We will be interested in the question, which tournaments may be represented as the dominance graph of a set of n coins.

Up to this point we made one, inessential simplification: we assume that x_i cannot be zero or infinity, that is, no coin can land on the same side with probability 1. If we have such a coin, we may replace it with a coin of type $(a, a, 1)$, that is, a fair coin that has the same number written on both sides. Since we are interested only in dominance graphs as tournaments we will also require that *for each pair of coins one dominates the other*. After fixing the parameters a_i and b_i for each coin, this restriction will exclude the points of $\binom{n}{2}$ hypersurfaces of codimension one from the set of possible values of $(x_1, x_2, \ldots, x_n)$, each hypersurface being defined by an equation involving a pair of variables $\{x_i, x_j\}$. The equations defining these surfaces may be obtained by replacing the inequality symbols with equal signs in the inequalities stated in Table 1. In particular, we assume that *different coins have different types*. In the rest of this section we will describe in terms of the types when the ith coin dominates the jth coin. To reduce the number of cases to be considered we first show that we may assume that no coin has the same number written on both sides. This is a direct consequence of the next lemma.

Lemma 2.1. *Suppose there is a coin of type (a_i, b_i, x_i), satisfying $a_i = b_i$. Replace this coin with a coin of type $(a_i', b_i, 1)$ where a_i' is any real number that is less than a_i but larger than any element in the intersection of the set $\{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\}$ with the open interval $(-\infty, a_i)$. Then modified system of coins has the same dominance graph.*

Proof. We only need to verify that another coin j, of type (a_j, b_j, x_j) dominates a coin of type (a_i, a_i, x_i) if and only if it dominates a coin of type $(a_i', a_i, 1)$. This is certainly the case when neither of a_j and b_j is equal to a_i, as these numbers compare to a_i the same way as to a_i'. We are left to consider the case when exactly one of a_j and b_j equals a_i (both cannot equal because then there is no directed edge between i and j in the dominance graph).

Case 1: $a_j = a_i$ and so $a_i' < a_i = a_j < b_j$ hold. In the original system, as well as in the modified one, only coin j can win against coin i and it does so with a positive probability. We have $j \to i$ in both dominance graphs.

Download English Version:

https://daneshyari.com/en/article/4646777

Download Persian Version:

https://daneshyari.com/article/4646777

Daneshyari.com