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Colorings of hypergraphs with large number of colors

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ABSTRACT

The paper deals with the well-known problem of Erdős and Hajnal concerning colorings of uniform hypergraphs and some related questions. Let $m(n,r)$ denote the minimum possible number of edges in an n-uniform non-r-colorable hypergraph. We show that for $r > n$,

$$c_1 \frac{n}{\ln n} \leqslant \frac{m(n,r)}{r^n} \leqslant C_1 n^3 \ln n,$$

where $c_1, C_1 > 0$ are some absolute constants. Moreover, we obtain similar bounds for $d(n,r)$, which is equal to the minimum possible value of the maximum edge degree in an n-uniform non-r-colorable hypergraph. If $r > n$, then

$$c_2 \frac{n}{\ln n} \leqslant \frac{d(n,r)}{r^{n-1}} \leqslant C_2 n^3 \ln n,$$

where $c_2, C_2 > 0$ are some other absolute constants.

1. Introduction

The paper deals the classical extremal combinatorial problem of P. Erdős and A. Hajnal concerning colorings of hypergraphs. Let us recall some definitions.

A *vertex coloring* of a hypergraph $H = (V, E)$ is a mapping $f : V \to \mathbb{N}$. Coloring f is said to be *proper* for H if there are no monochromatic edges in this coloring. A hypergraph is called *r-colorable* if there is a proper coloring with r colors for it. *The chromatic number* of the hypergraph H, $\chi(H)$, is the least r such that H is r-colorable, i.e. the minimum number of colors required for a proper coloring of H.

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1.1. Erdős–Hajnal problem

In 1961 P. Erdős and A. Hajnal proposed (see [7]) to determine the value $m(n, r)$ equal to the minimum possible number of edges in an n-uniform non-r-colorable hypergraph. Formally,

$$m(n, r) = \min\{|E| : H = (V, E) \text{ is } n\text{-uniform}, \ \chi(H) > r\}.$$

This problem, especially its 2-coloring case (Property B problem), has played a significant role in the development of probabilistic methods in combinatorics.

It is to easy to see that $m(n, r)$ is finite and satisfies the inequality

$$m(n, r) \leqslant \binom{r(n-1)+1}{n}. \tag{1}$$

For the graph case, $n = 2$, the bound (1) gives the exact answer. However in 1963–1964 Erdős showed (see [5,6]) that for hypergraphs the behavior of $m(n, 2)$ is quite different. By using probabilistic approach he proved that

$$2^{n-1} \leqslant m(n, 2) \leqslant \frac{e}{4} n^2 2^n \ln 2 \left(1 + O\left(\frac{1}{n}\right)\right).$$

Similar estimates for $m(n, r)$ obtained by the same way are the following (e.g., see [13]):

$$r^{n-1} \leqslant m(n, r) \leqslant \frac{e}{2} n^2 r^n \ln r \left(1 + O\left(\frac{1}{n}\right)\right). \tag{2}$$

The improvement of the lower bound in (2) has a long history (the reader is referred to the survey [13] for the details). The best current estimates for constant number of colors and large uniformity parameter were obtained by J. Radhakrishnan and A. Srinivasan (see [12]) for two colors, $r = 2$,

$$m(n, 2) = \Omega\left(\left(\frac{n}{\ln n}\right)^{\frac{1}{2}} 2^n\right),$$

and by D. Cherkashin and J. Kozik (see [4]) for $r > 2$,

$$m(n, r) = \Omega\left(\left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} r^{n-1}\right). \tag{3}$$

In the current paper we study the Erdős–Hajnal problem in the case when r is large and n small. It is easy to see that for fixed n and growing r, the simple bound (1) becomes better than (2) since its order of magnitude under these conditions is r^n, not $r^n \ln r$. However N. Alon showed in [1] that even for $r \gg n$, the estimate (1) is far away from the right answer. He proved that if $n \to \infty$ and $r/n \to \infty$ then

$$m(n, r) = O\left(n^{5/2} (\ln n) \left(\frac{3}{4}\right)^n \binom{r(n-1)+1}{n}\right) = O\left(n^2 (\ln n) \left(\frac{3e}{4}\right)^n r^n\right). \tag{4}$$

The first result of the paper refines Alon's result (4) as follows.

Theorem 1. *Suppose $r > n$. Then*

$$m(n, r) = O\left(n^{7/2} (\ln n) \left(\frac{1}{e}\right)^n \binom{r(n-1)+1}{n}\right) = O\left(n^3 (\ln n) r^n\right). \tag{5}$$

The bound (5) improves (2) for $\ln r = \Omega(n \ln n)$.

The first lower bound for $m(n, r)$ of order r^n for large values of r (note that the bounds (2), (3) give only r^{n-1}) was also obtained by Alon in [1]. He showed that

$$m(n, r) > (n-1) \left\lceil \frac{r}{n} \right\rceil \left\lfloor \frac{n-1}{n} r \right\rfloor^{n-1}, \tag{6}$$

which for $r > n$, implies that $m(n, r) = \Omega(r^n)$. A better result can be established by the help of a criterion for r-colorability of an arbitrary hypergraph in terms of so-called *ordered r-chains* proved by A. Pluhár in [11] (see Section 2 for the details). By using this criterion and some additional observation concerning the number of ordered r-chains Shabanov (see [13]) showed that for $r > n$,

$$m(n, r) = \Omega\left(n^{1/2} r^n\right). \tag{7}$$