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Planar graphs without adjacent cycles of length at most five are (1, 1, 0)-colorable

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ABSTRACT

Let $d_1, d_2, \ldots, d_k$ be k non-negative integers. A graph G is $(d_1, d_2, \ldots, d_k)$-colorable, if the vertex set of G can be partitioned into subsets $V_1, V_2, \ldots, V_k$ such that the subgraph $G[V_i]$ induced by V_i has maximum degree at most d_i for $i = 1, 2, \ldots, k$. Borodin, Montassier and Raspaud asked: Is every planar graph without adjacent cycles of length at most five 3-colorable, i.e., (0, 0, 0)-colorable? This problem has now been answered negatively by Cohen-Addad et al. who successfully constructed a non-3-colorable planar graph with neither 4-cycles nor 5-cycles. Is every planar graph without adjacent cycles of length at most five (1, 0, 0)-colorable? To this new problem, this paper proves that every planar graph without adjacent cycles of length at most five is (1, 1, 0)-colorable.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. For used but undefined terminology and notation we refer the reader to the book by Bondy and Murty [2].

Let $d_1, d_2, \ldots, d_k$ be k non-negative integers. A $(d_1, d_2, \ldots, d_k)$-*coloring* of a graph $G = (V, E)$ is a mapping from the vertex set V to a set of colors, say $\{1, 2, \ldots, k\}$, such that every vertex colored i has at most d_i neighbors colored i for $i = 1, 2, \ldots, k$.

Clearly, $(d_1, d_2, \ldots, d_k)$-coloring generalizes the classical proper coloring of graphs. The well-known Four Color Theorem says that every planar graph is (0, 0, 0, 0)-colorable. As for (0, 0, 0)-colorability of planar graphs, it is just the classical 3-colorability of planar graphs, which was proved to be an NP-complete problem by Garey et al. [15]. Let C_k denote a cycle of length k. The famous Three Color Theorem [16] says that every C_3-free planar graph is 3-colorable. Since K_4, the complete graph of order 4, is planar and not 3-colorable, it is easy to see that, for $k \geq 5$, the condition C_k-free does not guarantee the 3-colorability of planar graphs. Using K_4 and *quasi-edge*, one can easily construct a C_4-free planar graph, which is not 3-colorable, see [11]. On the other hand, all known non-3-colorable graphs contain 4-cycles as well as 5-cycles. Based on this, Steinberg proposed the following pregnant conjecture:

Steinberg's Conjecture ([2,25]). *Every planar graph with neither 4-cycles nor 5-cycles is 3-colorable.*

Progression towards Steinberg's conjecture had been made in different directions. The first direction is pointed by Erdös who suggested to find a constant C such that a planar graph without cycles of length from 4 to C is 3-colorable [25]. Abbott and Zhou [1] proved that such a constant C does exist, and $C \leq 11$. This result was later on improved to $C \leq 10$ by

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Borodin [4]; to $C \leq 9$ by Borodin [3] and independently by Sanders and Zhao [24]; and to $C \leq 7$ by Borodin et al. [8]. It is unknown whether $C \leq 6$.

The second direction started from a paper by Chang et al. [12] where the $(d_1, d_2, \ldots, d_k)$-colorings are called near colorings, and the authors proved that all planar graphs with neither 4-cycles nor 5-cycles are (4, 0, 0)- and (2, 1, 0)-colorable. This initial result has now been improved to the following theorem.

Theorem 1. *Every planar graph with neither 4-cycles nor 5-cycles is (3, 0, 0)-, (1, 1, 0)-, (2, 0, 0)-colorable, see [17,18,31,13], respectively.*

In studying 3-colorability of planar graphs, a new interesting trend in the literature has been appeared. The started point of the trend is a paper of Borodin and Raspaud [11], where the authors proved that every planar graph with neither 5-cycles nor triangles at distance less than 4 is 3-colorable, and proposed the following Bordeaux conjecture:

Bordeaux conjecture. (1) *Every planar graph with neither 5-cycles nor intersecting triangles is 3-colorable (the weak version).*
(2) *Every planar graph with neither 5-cycles nor adjacent triangles is 3-colorable (the strong version).*

Note that, if the strong Bordeaux conjecture is true, then it implies its weak version and the Steinberg's conjecture. Below are some known results motivated by Bordeaux conjecture:

Theorem 2. (1) *Every planar graph with neither 5-cycles nor triangles at distance 3 is 3-colorable [5,29];*
(2) *Every planar graph with neither 5-cycles nor triangles at distance 2 is 3-colorable [6];*
(3) *Every planar graph with neither 5-cycles nor adjacent triangles is (1, 1, 1)-colorable [30];*
(4) *Every planar graph without 5-cycles is (1, 1, 1)-colorable [28];*
(5) *Every planar graph with neither 5-cycles nor intersecting triangles is (1, 1, 0)-colorable [22];*
(6) *Every planar graph with neither 5-cycles nor intersecting triangles is (2, 0, 0)-colorable [21];*
(7) *Every planar graph with neither 5-cycles nor adjacent triangles is (1, 1, 0)-colorable [19].*

It seems that no special reason supports the lacking of 5-cycles to guarantee the 3-colorability of planar graphs. Allowing presence of cycles of any length, what can guarantee the 3-colorability of planar graphs? Borodin with his coauthors proposed a challenging problem and an ultimate conjecture as follows (which can be viewed as new engines to push the study of 3-colorability of planar graphs forward).

Problem 1 (*[9]*)**.** Is every planar graph without adjacent cycles of length at most 5 3-colorable?

Nsk's conjecture (*[7]*)**.** *Every planar graph with neither triangular 3-cycles nor triangular 5-cycles is 3-colorable.*

Clearly Problem 1, as a relaxation of the Nsk's conjecture, is still stronger than the strong Bordeaux conjecture. Earlier related papers to Problem 1 or Nsk's conjecture may be summarized in the following theorem.

Theorem 3. (1) *Every planar graph without triangular 9^--cycles is 3-colorable [7];*
(2) *Every planar graph without adjacent 7^--cycles is 3-colorable [10,26];*
(3) *Every planar graph without triangular cycles of length from 4 to 7 is 3-colorable [8];*
(4) *Every planar graph without 5^--cycles at distance less than 4 is 3-colorable [23];*
(5) *Every planar graph without 5^--cycles at distance less than 2 is 3-colorable [20].*

Very recently, Cohen-Addad et al. [14] surprisingly disproved Steinberg's conjecture by constructing a non-3-colorable planar graph with neither 4-cycles nor 5-cycles. Since Nsk's conjecture, as well as Problem 1, is stronger than the strong Bordeaux conjecture, while the later is stronger than Steinberg's conjecture, all these appealing conjectures are disproved by [14]! Nevertheless, One may asked many relaxed problems, one of them may be as follows:

Problem 2. Is every planar graph without adjacent cycles of length at most five is (1, 0, 0)-colorable?

In this paper, we will prove the following result.

Theorem 4. *Every planar graph without adjacent cycles of length at most five is (1, 1, 0)-colorable.*

This clearly provides a partial solution to Problem 2 and directly improves one result in Theorem 1.

The rest of this section is devoted to some definitions. A graph G is *planar* if it can be embedded into the plane so that its edges meet only at their ends. Any such particular embedding of a planar graph is called a *plane* graph. For a plane graph G, we use V, E and F to denote its vertex set, edge set and face set, respectively. For a vertex $v \in V$, the degree of v in G, denoted by $d_G(v)$, or simply $d(v)$, is the number of edges incident with v in G. The neighborhood of v in G, denoted $N_G(v)$, or simply $N(v)$, consists of all vertices adjacent to v in G. Call v a *k-vertex*, or a *k^+-vertex*, or a *k^--vertex*, if $d(v) = k$, or $d(v) \geq k$, or $d(v) \leq k$, respectively. For a face $f \in F$, the number of edges on the boundary of f (where cut edge, if any, is counted twice), denoted $d(f)$, is called the *degree* of f. Analogously, the notations above for vertices will be applied to faces or cycles, too. We write $f = [v_1v_2 \ldots v_k]$ if $v_1, v_2, \ldots, v_k$ are consecutive vertices on f in a cyclic order, and say that f is a *$(d(v_1), d(v_2), \ldots, d(v_k))$-face*.