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Planar graphs without 4-cycles adjacent to triangles are 4-choosable

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ABSTRACT

Lam et al. proved that every planar graph without 4-cycles is 4-choosable (Lam et al., 1999). In this paper, we improve this result by showing that every planar graph without 4-cycles adjacent to triangles is 4-choosable.

1. Introduction

All graphs considered are finite, simple and undirected. For used but undefined terminology and notation in this paper, we refer the reader to the book by Bondy and Murty [1].

A *list assignment* of a graph $G = (V, E)$ is a function L that assigns to each vertex $v \in V$ a list $L(v)$ of colors. An *L-coloring* of G is a function $\lambda : V \longrightarrow \cup_{v \in V} L(v)$ such that $\lambda(v) \in L(v)$ for every $v \in V$ and $\lambda(u) \neq \lambda(v)$ whenever $uv \in E$. If G admits an L-coloring, then it is *L-colorable*. A graph G is *k-choosable* if it is L-colorable for every list assignment L with $|L(v)| \geq k$ for every $v \in V$.

All 2-choosable graphs have been characterized by Erdős et al. [2]. Thomassen [7] proved that every planar graph is 5-choosable. Voigt [8] showed that not all planar graphs are 4-choosable. Gutner [5] proved that the problems to determine whether a given planar graph is 3- or 4-choosable are NP-hard. So nice sufficient conditions for a planar graph to be 3- or 4-choosable are of certain interest.

This paper only concerns 4-choosability of planar graphs. Two cycles are *intersecting* if they have at least one vertex in common; *adjacent* if they have at least one edge in common. Let k be a positive integer. A cycle of length k is called a *k-cycle*. A 3-cycle is usually called a *triangle*. A cycle in G is called *triangular*, if it is adjacent to a triangle. Call a graph G *k-degenerate* if every subgraph of G has at least one vertex of degree k. Clearly, every k-degenerate graph is $(k + 1)$-choosable. In particular, every 3-degenerate graph is 4-choosable. By a simple discharging argument, it is easy to see that every planar graph without triangles is 3-degenerate, hence 4-choosable. Up to date, the 4-choosability of planar graphs without k-cycles proved non-trivially only for $k = 4, 5, 6, 7$. More precisely, every planar graph without 5-cycles [9], or without 6-cycles [4], is 3-degenerate, hence 4-choosable. It is interesting to notice that, for $k = 4, 7$, there are planar graphs without k-cycles which are not 3-degenerate. So the 4-choosability of planar graphs without 4-cycles [6], or without 7-cycles [3], is established not via 3-degeneracy. Although it is appealing to determine the set of all positive integers K such

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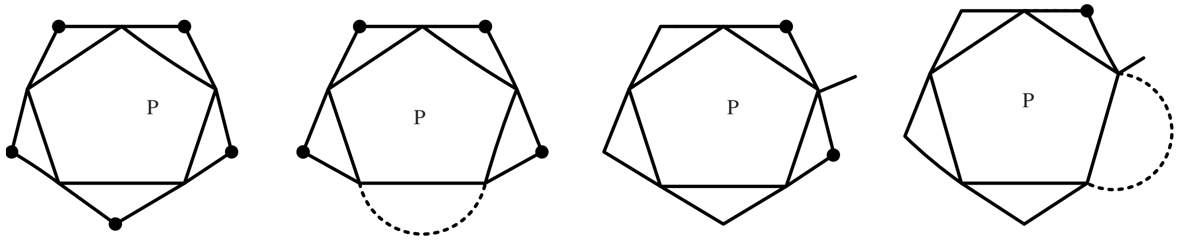

Fig. 1. A sink P with its sources in G.

that, for every $k \in K$, every planar graph without k-cycles is 4-choosable, it seems very difficult to attain. Are there any other nice sufficient conditions for a planar graph to be 4-choosable? Without intersecting short cycles, or more strongly, without adjacent short cycles might be good choice. Theorem A is the first example of such choice.

Theorem A. *A planar graph is* 4*-choosable if it does not contain intersecting triangles* [10].

Conjecture B. *Every planar graph without adjacent triangles is* 4*-choosable* [10,6].

Nearby Conjecture B, this paper proves a result as follows:

Theorem 1. *Every planar graph without triangular* 4*-cycles is* 4*-choosable.*

2. Structures

Suppose to the contrary that Theorem 1 is not true. Let $G = (V, E)$ be a counterexample to Theorem 1 with the fewest vertices. Then G clearly has the following elementary properties:

(1) G has no triangular 4-cycle.
(2) G is connected.
(3) G is not 4-choosable, i.e., there is a list assignment $L = \{L(v) \| L(v)| \geq 4, \forall\, v \in V\}$ such that G is not L-colorable. However,
(4) any proper vertex-induced subgraph G' of G is 4-choosable. In particular, G' is L'-colorable, where L' is the restriction of L on G'.

Embedding G into the plane, we get a plane graph $G = (V, E, F)$, where V, E and F are the set of vertices, edges, and faces of G, respectively. For a vertex $v \in V$, the *degree* of v, denoted $d(v)$, is the number of edges incident with v in G. For a face $f \in F$, the *degree* of f, denoted $d(f)$, is the number of edges incident with f (a cut-edge is counted twice). A vertex $v \in V$ is called a k-, k^+, or k^-*-vertex* if $d(v) = k$, $\geq k$, or $\leq k$, respectively. The notion of a k-, k^+-, or k^--face is similarly defined. For a face $f \in F$, if the vertices on f in a cyclic order are $v_1, v_2, \ldots, v_k$, then we write $f = [v_1 v_2 \ldots v_k]$, and call f a $(d(v_1), d(v_2), \ldots, d(v_k))$*-face.*

Using the minimality of G, i.e., the elementary property (3) and (4) above, the following lemma is straightforward.

Lemma 1 ([6]). *G has no 3^--vertex.* □

Two faces in G are *adjacent* if they have at least one edge in common; *normally* adjacent if they are adjacent and have exactly two vertices in common. Call a 5-face *bad* if it is adjacent to at least four 3-faces, and incident with either five 4-vertices or four 4-vertices and one 5-vertex; *good* otherwise. Note that if a 5-face is adjacent to at most three 3-faces, or incident with one 6^+-vertex or two 5^+-vertices, then it is good.

Let P be a 5-face and T a 3-face in G. If P and T are adjacent, then they are normally adjacent since G has no triangular 4-cycles. The vertex v on T but not on P is called a *source* of P, if all of the following hold:

- P is a bad 5-face,
- if P has a 5-vertex u, then $uv \in T$.

If v is a source of P, then P is equivalently called a *sink* of v. Clearly, for $k = 4, 5$, a bad 5-face P with no 5-vertex has k sources if and only if it is adjacent to k 3-faces. Suppose a bad 5-face P is incident with one 5-vertex. Then it has one or two sources depending on the 5-vertex being incident with one or two 3-faces, respectively, see Fig. 1, where black points represent sources of P.

A face $f \in F$ is *simple* if its boundary $b(f)$ is a cycle.

Lemma 2. *Every 5^--face in G is simple.*