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On the independence ratio of distance graphs

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ABSTRACT

A distance graph is an undirected graph on the integers where two integers are adjacent if their difference is in a prescribed distance set. The *independence ratio* of a distance graph G is the maximum density of an independent set in G. Lih et al. (1999) showed that the independence ratio is equal to the inverse of the fractional chromatic number, thus relating the concept to the well studied question of finding the chromatic number of distance graphs.

We prove that the independence ratio of a distance graph is achieved by a periodic set, and we present a framework for discharging arguments to demonstrate upper bounds on the independence ratio. With these tools, we determine the exact independence ratio for several infinite families of distance sets of size three and determine asymptotic values for others.

1. Introduction

For a set S of positive integers, the *distance graph* $G(S)$ is the infinite graph with vertex set $\mathbb{Z}$ where two integers i and j are adjacent if and only if $|i-j| \in S$. Intense study of distance graphs began when Eggleton, Erdős, and Skilton [15,16] defined them as a modified version of the Hadwiger–Nelson problem of coloring the unit-distance graph on $\mathbb{R}^2$. The chromatic number of distance graphs has since been widely studied [2,5,8–11,13,14,19,15–18,22,24–27,29,31–37,40–42].

A particularly effective tool for finding lower bounds on the chromatic number is to determine the *fractional chromatic number*, $\chi_f(S) = \chi_f(G(S))$. A *fractional coloring* of a graph G is a function c from the independent sets I of G to nonnegative real numbers such that for every vertex v, the sum $\sum_{I \ni v} c(I) \geq 1$, and the *value* of the coloring is the sum $\sum_I c(I)$ taken over all independent sets I. The fractional chromatic number $\chi_f(G)$ is the minimum value of a fractional coloring, and provides a lower bound on the chromatic number. In fact, $\lceil \chi_f(G) \rceil \leq \chi(G)$ and this inequality is frequently sharp in the case of distance graphs [8–10,29,42].

For an independent set A in $G(S)$ the density $\delta(A)$ is equal to $\limsup_{N\to\infty} \frac{|A\cap[-N,N]|}{2N+1}$. The *independence ratio* $\overline{\alpha}(S)$ is the supremum of $\delta(A)$ over all independent sets A in $G(S)$. Lih, Liu and Zhu showed that determining the fractional chromatic number $\chi_f(S)$ of a distance graph $G(S)$ is equivalent to determining its independence ratio.

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Theorem 1 (*Lih, Liu, and Zhu [31]*). *Let S be a finite set of positive integers. Then $\chi_f(S) = \overline{\alpha}(S)^{-1}$.*

Thus, the previous results computing the fractional chromatic number of distance graphs apply to the independence ratio. We summarize these results in Section 3. The inequality $\chi_f(S) \leq \overline{\alpha}(S)^{-1}$ is easy to determine if you have access to a *periodic* independent set in $G(S)$ of density $\overline{\alpha}(S)$. While this was acknowledged by previous work, it is surprising that no one proved that such an independent set exists. Thus, we demonstrate that such periodic extremal sets exist.

Theorem 2. *Let S be a finite set of positive integers and let $s = \max S$. There exists a periodic independent set A in $G(S)$ with period at most $s2^s$ where $\delta(A) = \overline{\alpha}(S)$.*

Our proof of Theorem 2 uses a lemma (Lemma 3) about extremal walks in finite digraphs that we also apply to show there exist periodic extremal sets for dominating sets and identifying codes in finitely-generated distance graphs. This lemma may be also applicable in other situations.

We also develop several fundamental techniques for determining the independence ratio, which we then apply to several infinite families of distance sets of size three. To prove upper bounds on $\overline{\alpha}(S)$, we develop a new discharging method. The resulting *Local Discharging Lemma* (Lemma 21) is then used extensively to give exact values of $\overline{\alpha}(S)$ for several infinite families of distance sets. We witness several common themes among these proofs, and these themes may be evidence that discharging arguments of this type could be used to determine almost all values of $\overline{\alpha}(S)$.

For an integer n, the *circulant graph* $G(n, S)$ is the graph whose vertices are the integers modulo n where two integers i and j are adjacent if and only if $|i-j| \equiv k \pmod{n}$, for some $k \in S$. The distance graph $G(S)$ can be considered to be the limit structure of the sequence of circulant graphs $G(n, S)$. Thus, extremal questions over the circulant graphs lead to extremal questions on the distance graph. For instance, the equality $\overline{\alpha}(S) = \limsup_{n\to\infty} \alpha(G(n, S))/n$ is a consequence of Theorem 1.

Since the complement of a circulant graph is also a circulant graph, and the independence number of a graph is the clique number of its complement, studying the independence ratio of distance graphs is strongly related to determining the independence number and clique number of circulant graphs. In particular, simultaneously bounding the independence number and clique number of circulant graphs has shown lower bounds on Ramsey numbers [30]. So far, these parameters have been studied for circulant graphs $G(n, S)$ when limited to special classes of sets S, whether algebraically defined [1,4,6,12,20,28,39] or with S finite and n varying [3,7,23].

A recent development is the discovery that certain circulant graphs $G(n, S)$ are *uniquely K_r-saturated*, including three infinite families [21]. A graph H is uniquely K_r-saturated if H contains no copy of K_r and for every pair $uv \notin E(H)$ there is a unique copy of K_r in $H + uv$. The first step in proving this property is showing that the clique number of $G(n, S)$ is equal to $r - 1$. In the three infinite families, the generating set S uses a growing number of elements, but the complement of the graph uses a finite number of elements. The complement $\overline{G(n, S)}$ is another circulant graph $G(n, S')$ where S' has a finite number of elements. The independence number of $G(n, S')$ is of particular interest. Our discharging method is an adaptation of the discharging method used in [21] to determine the independence number in circulant graphs.

We start in Section 2 by proving Theorem 2, that the independence ratio in a distance graph is achievable by a periodic independent set. We take the opportunity there to show two quick applications of the proof technique to related density problems for other types of subset of $G(S)$. In Section 3 we summarize previous results on the independence ratio. The next section collects some introductory results concerning $\overline{\alpha}(S)$, and then in Section 5 we define our discharging process and its connection to the independence ratio, proving the Local Discharging Lemma. We use the Local Discharging Lemma to prove exact values of $\overline{\alpha}(S)$ for several families of sets S in Section 6. We determine the independence ratio for a range of graphs with generator sets of size 3. This extends work of Zhu [42], who considered the fractional chromatic number of $G(S)$ in distance graphs with generator sets of size 3. Finally in Section 7 we discuss the algorithm we used to compute values of $\overline{\alpha}(S)$ for specific finite sets S. The computed values of $\overline{\alpha}(\{1, 1+k, 1+k+i\})$ are given as a table in Appendix A, while more values are given in data available online.[1]

Our notation is standard. Throughout the paper we consider S to be a finite, nonempty set of positive integers. For a positive integer n, we write $[n] = \{1, \dots, n\}$, and similarly $[a, b] = \{a, a+1, \dots, b-1, b\}$. When $d \geq 1$, we let $d \cdot S = \{d \cdot s : s \in S\}$ and $S + d = \{s + d : s \in S\}$.

2. Periodic sets of extremal value

In this section, we prove Theorem 2. As a consequence, we give an alternative proof of Theorem 1. We start by proving a lemma that implies that periodic extremal sets exist for several different extremal problems on distance graphs, such as dominating sets and r-identifying codes.

Consider a finite directed graph G where every vertex v is given a weight $w(v)$. Let $W = (v_i)_{i\in\mathbb{Z}}$ be a doubly infinite walk on G. Then the *upper average weight* $\overline{w}(W)$ of W is defined as $\limsup_{N\to\infty} \frac{\sum_{i=-N}^{N} w(v_i)}{2N+1}$, and the *lower average weight* $\underline{w}(W)$ of W is defined as $\liminf_{N\to\infty} \frac{\sum_{i=-N}^{N} w(v_i)}{2N+1}$. Given a simple cycle C in G, define the infinite walk W_C by infinitely repeating C. Observe that $\overline{w}(W_C) = \underline{w}(W_C) = \frac{\sum_{v\in V(C)} w(v)}{|C|}$.

[1] See http://www.math.iastate.edu/dstolee/r/distance.htm for all data files.