Discrete Mathematics 339 (2016) 3085–3088

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Vertex-disjoint copies of $K_{1,3}$ in $K_{1,r}$-free graphs[☆]

Suyun Jiang, Jin Yan [*]

School of Mathematics, Shandong University, Jinan 250100, China

ARTICLE INFO

Article history:
Received 7 December 2015
Received in revised form 11 June 2016
Accepted 18 June 2016
Available online 20 July 2016

Keywords:
$K_{1,r}$-free graph
Vertex-disjoint stars
Minimum degree

ABSTRACT

A graph G is said to be $K_{1,r}$-free if G does not contain an induced subgraph isomorphic to $K_{1,r}$. Let k, r be integers with $k \geq 2$, $r \geq 4$. In this paper, we prove that if G is a $K_{1,r}$-free graph of order at least $(k-1)(3r-2)+1$ with $\delta(G) \geq 3$, then G contains k vertex-disjoint copies of $K_{1,3}$. This result shows that Fujita's conjecture (2008) is true for $t = 3$ and $r \geq 4$.

1. Introduction

We discuss only finite simple graphs and use standard terminology and notation from [1] except as indicated. For a graph G, we denote by $V(G)$, $E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of G, respectively. We use E to denote the edge set of G if there is no confusion. A set of subgraphs of G is said to be disjoint if no two of them have any vertex in common. For disjoint subgraphs H_1 and H_2 of G, the union of H_1 and H_2, denoted by $H_1 \cup H_2$, is the subgraph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. By starting with $H_1 \cup H_2$ and adding edges joining every vertex of H_1 to every vertex of H_2, we obtain the join of H_1 and H_2, denoted by $H_1 + H_2$. A graph G is said to be $K_{1,r}$-free if G does not contain an induced subgraph isomorphic to $K_{1,r}$. In particular, a graph G is said to be claw-free when G is $K_{1,3}$-free. Let K_n be a complete graph of order n.

Sumner [7] showed that a connected claw-free graph of order $2k$ contains a perfect matching, i.e., k disjoint copies of K_2. Note that K_2 is $K_{1,1}$, Fujita considered the existence of k disjoint copies of $K_{1,t}$ $(t \geq 2)$ in forbidden graphs. In [3,4], Fujita proposed the following conjecture:

Conjecture 1.1 (*Fujita, [3,4]*). *Let k, r, t be integers with $k \geq 2$, $r \geq 3$ and $t \geq 2$. If G is a $K_{1,r}$-free graph of order at least $(k-1)(t(r-1)+1)+1$ with $\delta(G) \geq t$, then G contains k disjoint copies of $K_{1,t}$.*

If the conjecture is true, the bound on $|V(G)|$ is best possible. To see this, let $B_i = K_t$ for each i with $1 \leq i \leq r-1$, and consider $G = \cup_{i=1}^{k-1} A_i$, where $A_i = K_1 + \cup_{j=1}^{r-1} B_j$ for each i with $1 \leq i \leq k-1$. Then G is a $K_{1,r}$-free graph of order $(k-1)(t(r-1)+1)$ with $\delta(G) \geq t$. It is easy to check that G does not contain k disjoint copies of $K_{1,t}$.

Fujita [3] confirmed that the conjecture is true for $t = 2$, and proved the following theorem in [4], which shows that Conjecture 1.1 is true for $t = r = 3$ because $K_1 + (K_1 \cup K_2)$ contains $K_{1,3}$.

Theorem 1.2 (*Fujita, [4]*). *Let k be an integer with $k \geq 2$. If G is a claw-free graph of order at least $7k-6$ with $\delta(G) \geq 3$, then G contains k disjoint copies of $K_1 + (K_1 \cup K_2)$.*

☆ This work is supported by National Natural Science Foundation of China (No. 11271230).
* Corresponding author.
E-mail address: yanj@sdu.edu.cn (J. Yan).

http://dx.doi.org/10.1016/j.disc.2016.06.015
0012-365X/

Recently, Gao and Zou proved a similar result for $K_{1,4}$-free graph.

Theorem 1.3 (*Gao and Zou, [5]*). *Let $k \geq 2$ be an integer. If G is a $K_{1,4}$-free graph of order at least $11k - 10$ with $\delta(G) \geq 4$, then G contains k disjoint copies of $K_1 + (K_1 \cup K_2)$.*

In this paper, we prove the following result, combined with Theorem 1.2, we see that Conjecture 1.1 is true for $t = 3$.

Theorem 1.4. *Let k, r be integers with $k \geq 2$, $r \geq 4$. If G is a $K_{1,r}$-free graph of order at least $(k-1)(3r-2)+1$ with $\delta(G) \geq 3$, then G contains k disjoint copies of $K_{1,3}$.*

There are some results concerning the existence of k disjoint copies of K_3 in forbidden graphs. Wang [8] proved that if G is a claw-free graph of order at least $6k - 5$ with $\delta(G) \geq 3$, then G contains k disjoint copies of K_3. In the same paper, Wang proposed the following conjecture: For each integer $t \geq 4$, there exists an integer k_t depending on t only such that $h(t, k) = 2t(k - 1)$ for all integers $k \geq k_t$, where $h(t, k)$ is the smallest integer m such that every $K_{1,t}$-free graph of order greater than m and with minimum degree at least t contains k disjoint triangles. However, in [9], Zhang et al. totally disproved the conjecture and obtained a lower bound and an upper bound of $h(t, k)$.

Ramsey number is a very useful tool in this paper. For graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest positive integer n such that every graph G of order at least n contains G_1 or the complement of G contains G_2. The following is a well-known result of Chvátal [2].

Theorem 1.5 (*Chvátal, [2]*). *Let T_n be a tree of order n. Then $R(T_n, K_m) = (n - 1)(m - 1) + 1$.*

In particular, we have the following Corollary (also see [6] in Page 26).

Corollary 1.6. $R(K_{1,n}, K_m) = n(m - 1) + 1$.

We use the following notations in this paper. For a subset U of $V(G)$, $G[U]$ denotes the subgraph of G induced by U. If H is a subgraph of G, written as $G \supseteq H$, and let $G - H = G[V(G) - V(H)]$. For a subgraph H of G and a vertex $x \in V(G)$, the neighborhood of x in H is denoted by $N(x, H)$ and let $d(x, H) = |N(x, H)|$. For disjoint subgraphs H_1 and H_2 of G, we let $E(H_1, H_2)$ denote the set of edges of G joining a vertex in H_1 and a vertex in H_2, and $N(H_1, H_2)$ denote the set of neighbors of H_1 in H_2. Clearly, $|N(H_1, H_2)| = |\cup_{v \in H_1} N(v, H_2)| \leq \sum_{v \in H_1} d(v, H_2)$.

2. Proof of Theorem 1.4

Let k, r be integers with $k \geq 2$, $r \geq 4$. Let G be a $K_{1,r}$-free graph of order at least $(k-1)(3r-2)+1$ with $\delta(G) \geq 3$. Take s disjoint subgraphs $C_1, C_2, \ldots, C_s$ such that C_i contains $K_{1,3}$ as a spanning subgraph for each i with $1 \leq i \leq s$. Let $C = \cup_{i=1}^{s} C_i$ and $H = G - C$. We choose $C_1, C_2, \ldots, C_s$ so that

$$s \text{ is maximum,} \tag{1}$$

and subject to (1),

$$\sum_{i=1}^{s} |E(C_i)| \text{ is maximum.} \tag{2}$$

We may assume that $s \leq k - 1$. By the maximality of s, H does not contain a copy of $K_{1,3}$. Thus we have $\Delta(H) \leq 2$. Note that $\delta(G) \geq 3$, we see $d(v, C) \geq 1$ for each $v \in V(H)$. It follows that $|N(C, H)| = |H| \geq (k-1)(3r-2)+1-4s \geq (3r-6)s+1$ as $s \leq k - 1$. Note that $\sum_{i=1}^{s} |N(C_i, H)| \geq |N(C, H)|$, so there exists a C_i, say C_1, such that $|N(C_1, H)| \geq 3r - 5$.

Let $V(C_1) = \{a, b, c, d\}$ with $d(a, C_1) = 3$. By the maximality of s, we see $G[V(H \cup C_1)]$ does not contain two disjoint copies of $K_{1,3}$. We first prove the following claims.

Claim 2.1. *If $|N(x, H)| \geq 3$ for some $x \in V(C_1)$, we may assume that $\{x_1, x_2, x_3\} \subseteq N(x, H)$, then $|N(y, H - \{x_1, x_2, x_3\})| \leq 2$ for each $y \in V(C_1) - x$.*

Proof. If $|N(y, H - \{x_1, x_2, x_3\})| \geq 3$ for some $y \in V(C_1) - x$, then $G[\{x, x_1, x_2, x_3\}] \supseteq K_{1,3}$ and $G[N(y, H - \{x_1, x_2, x_3\}) \cup \{y\}] \supseteq K_{1,3}$, it follows that $G[V(H \cup C_1)]$ contains two disjoint copies of $K_{1,3}$, this is contrary to the maximality of s. □

Claim 2.2. *If $E(C_1) = \{ab, ac, ad, bc, bd\}$ and $|N(x, H)| \geq 4$ for some $x \in \{a, b\}$, then $|N(y, H)| \leq 1$ for each $y \in \{c, d\}$.*

Proof. Note that $d(a, C_1) = d(b, C_1) = 3$ and $d(c, C_1) = d(d, C_1) = 2$. We see that a and b are symmetric, and c and d are symmetric. We may assume $|N(a, H)| \geq 4$. If $|N(c, H)| \geq 2$ or $|N(d, H)| \geq 2$, by symmetry, we may assume $|N(c, H)| \geq 2$. It is easy to see that $G[N(c, H) \cup \{c, b\}]$ contains a copy of $G_1 \cong K_{1,3}$ such that $\{c, b\} \subseteq V(G_1)$. Thus $|N(a, H) - V(G_1)| \geq 2$. So we have $G[(N(a, H) - V(G_1)) \cup \{a, d\}] \supseteq K_{1,3}$, it follows that $G[V(H \cup C_1)]$ contains two disjoint copies of $K_{1,3}$, this is contrary to the maximality of s. □