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Cardinality of relations with applications

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ABSTRACT

Based on Y. Kawahara's characterisation of the cardinality of relations we derive some fundamental properties of cardinalities concerning vectors, points and mapping-related relations. As applications of these results we verify some properties of linear orders and graphs in a calculational manner. These include the cardinalities of rooted trees and some estimates concerning graph parameters. We also calculationally prove the result of D. Kőnig that in bipartite graphs the matching number equals the vertex cover number.

1. Introduction

Based on pioneering work of mainly G. Boole, A. De Morgan, C.S. Peirce and E. Schröder in the 19th century, the modern axiomatic investigations of the calculus of binary relations started with the seminal paper [28] of A. Tarski on relation algebra in the middle of the 20th century. Since the 1970s this algebraic structure has widely been used by many mathematicians, engineers and computer scientists as a conceptual and methodological base for problem solving in areas like graph theory, theory of orders and lattices, combinatorics, preference and scaling, social choice theory, algorithmics, data bases, and semantics of programming languages. A lot of examples and references to relevant literature can be found e.g., in [2,8,10,11,25,27] and the proceedings of the conference series "Relational and Algebraic Methods in Computer Science".

The use of relation algebra brings many advantages: Concerning modelling, it is mainly due to the fact that relations and many objects of discrete mathematics are essentially the same or closely related. For instance, a directed graph is nothing else than a relation on a non-empty and finite set of vertices, and also for other classes of graphs there are simple and elegant ways to model them with relations, as shown in [25,27], for example. Secondly, the use of relation algebra frequently leads to very precise proofs, where calculational transformations constitute the decisive parts. This has the advantage of clarifying the proof structure frequently, reducing the danger of doing wrong proof steps and to opening the possibility for proof mechanisation, for instance, by automated theorem provers or proof assistants. See e.g., [3,4,7,13,15] for the latter. Thirdly, the set-theoretic standard model of relation algebra can easily and efficiently be implemented. This supports prototyping and validation tasks in a significant manner, e.g., by the BDD-based special-purpose computer algebra system RelView (see [6]).

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Experience has also shown that for advanced applications the "classical" homogeneous relation algebra in the sense of [28] (and further developed in [19,20,29], for example) has to be modified. To be able to treat not only relations on one universe but on different sets, in [24] types have been introduced, leading to the notion of a heterogeneous relation algebra. Based on this and following the manner how K.C. Ng and A. Tarski added in [30] the Kleene star as an additional operation for reflexive–transitive closures to homogeneous relation algebra, relational products, sums and embeddings have been axiomatised to deal, for example, with n-ary functions, case distinctions and restrictions, respectively. Set-theoretic membership relations and some variants (on function domains) have also been introduced in the same way, mainly for the use in relation-algebraic semantics. See [2,25,27,31] for details and references to relevant literature.

In this paper we investigate an extension of heterogeneous relation algebra. We are concerned with a cardinality operation on relations, the axiomatisation of which originates from [16]. In this paper Y. Kawahara acknowledges the considerable influence of [27] to the formal relation-algebraic study of graphs. But he also mentions that "the cardinality of relations is treated rather implicitly or intuitively" [16, Page 251]. Therefore, he develops a cardinality operation on relations and demonstrates by some applications in basic graph theory that its axiomatic specification can be used to reason about cardinalities of relations in a purely calculational and algebraic manner. In [5] the axiomatisation of [16] is applied for the formal assertion-based development and verification of relational approximation algorithms, where cardinalities play an important role when proving the desired approximation bound.

The present paper is a continuation of [16,5]. We extend the stock of fundamental properties of cardinalities of relations by several results that concern vectors, points and mapping-related relations. In this regard the point axiom and the decomposition of relations into disjoint unions play an important role. To show the usefulness of the properties, we present some applications. The remainder of the paper is organised as follows: In Sections 2 and 3 we shortly recall those fundamentals of heterogeneous relation algebra we will need in the following sections; this includes the point axiom and some important consequences. Then, in Section 4, we present Y. Kawahara's axiomatisation of the cardinality operation on relations and some general properties. Specific properties of the cardinality operation with regard to vectors and points and of relations which are related to mappings are presented in Sections 5 and 6, respectively. Some simple applications that base on these properties are shown in Section 7, e.g., calculational proofs of cardinalities of rooted trees and of some estimates concerning well-known graph parameters. In Section 8 we apply our results to a more complex example. We calculationally prove the theorem of D. Kőnig saying that in bipartite graphs the matching number and the vertex cover number coincide. Section 9 contains some concluding remarks.

2. Relation-algebraic prerequisites

In this section we recall the fundamentals of relation algebra based on the heterogeneous approach of [24] and further developed especially in [25,27]. Set-theoretic relations form the standard model of relation algebras. We assume the reader to be familiar with the basic operations on them, viz. R^{T} (transposition), $\overline{R}$ (complementation), $R \cup S$ (union), $R \cap S$ (intersection), $R;S$ (composition), the predicates $R \subseteq S$ (inclusion) and $R = S$ (equality), and the special relations O (empty relation), L (universal relation) and I (identity relation). Relations of the same type equipped with the Boolean operations, the inclusion and the constants O and L form complete Boolean lattices. Some further well-known algebraic properties of relations are $\overline{R^{\mathsf{T}}} = \overline{R}^{\mathsf{T}}$, $(R \cup S)^{\mathsf{T}} = R^{\mathsf{T}} \cup S^{\mathsf{T}}$, $(R \cap S)^{\mathsf{T}} = R^{\mathsf{T}} \cap S^{\mathsf{T}}$, $(R^{\mathsf{T}})^{\mathsf{T}} = R$, $(R;S)^{\mathsf{T}} = S^{\mathsf{T}};R^{\mathsf{T}}$, and the monotonicity of transposition, union, intersection and composition.

The theoretical framework for these laws (and many others) to hold is that of a (heterogeneous) *relation algebra* in the sense of [24,25,27], with typed relations as elements. This implies that each relation has a source and a target and we write $R : X \leftrightarrow Y$ to express that R is of type $X \leftrightarrow Y$ with source X and target Y. In case of set-theoretic relations $R : X \leftrightarrow Y$ means that R is a subset of the direct product $X \times Y$ and then X and Y are also called *carrier sets*. As constants and operations of a relation algebra we have those of set-theoretic relations, where we frequently overload the symbols O, L and I, i.e., avoid the binding of types to them. Only when helpful or necessary we use indices to annotate types such as L_{XY} for the universal relation of type $X \leftrightarrow Y$ and I_X for the identity relation of type $X \leftrightarrow X$. The axiomatisation of relation algebra we will present now follows [25,27].

Axioms 2.1 (*Relation Algebra*)**.** The following hold:

(R1) For all types $X \leftrightarrow Y$ the relations of type $X \leftrightarrow Y$ constitute a complete Boolean lattice under the Boolean operations, the inclusion, the empty relation and the universal relation.
(R2) Composition of relations is associative and the identity relations are neutral elements with respect to composition.
(R3) For all relations Q, R and S (with appropriate types) the three inclusions $Q;R \subseteq S$, $Q^{\mathsf{T}};\overline{S} \subseteq \overline{R}$ and $\overline{S};R^{\mathsf{T}} \subseteq \overline{Q}$ are equivalent.
(R4) For all relations R and all universal relations (with appropriate types) from $R \neq \mathsf{O}$ it follows $\mathsf{L};R;\mathsf{L} = \mathsf{L}$.

In [27] the equivalences of (R3) are called the *Schröder rules* and the implication of (R4) is called the *Tarski rule*. In the relation-algebraic proofs of this paper we will mention only applications of (R3), (R4) and "non-obvious" consequences of the axioms, like the inclusion

$$Q;R \cap S \subseteq (Q \cap S;R^{\mathsf{T}});(R \cap Q^{\mathsf{T}};S), \qquad (1)$$