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Set partition patterns and statistics[☆]

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ABSTRACT

A set partition σ of $[n] = \{1, \ldots, n\}$ contains another set partition π if restricting σ to some $S \subseteq [n]$ and then standardizing the result gives π. Otherwise we say σ avoids π. For all sets of patterns consisting of partitions of [3], the sizes of the avoidance classes were determined by Sagan and by Goyt. Set partitions are in bijection with restricted growth functions (RGFs) for which Wachs and White defined four fundamental statistics. We consider the distributions of these statistics over various avoidance classes, thus obtaining multivariate analogues of the previously cited cardinality results. This is the first in-depth study of such distributions. We end with a list of open problems.

1. Introduction

There has been an explosion of papers recently dealing with pattern containment and avoidance in various combinatorial structures. And the study of statistics on combinatorial objects has a long and venerable history. By comparison, there are relatively few papers which study a variety of statistics on a number of different avoidance classes. The focus of the present work is pattern avoidance in set partitions combined with four important statistics defined by Wachs and White [10]. It is the first comprehensive study of these statistics on avoidance classes. In particular, we consider the distribution of these statistics over every class avoiding a set of partitions of $\{1, 2, 3\}$. We should note that there are other statistics on the family of all set partitions which have yielded interesting results. For example, there is a bijection between set partitions and rook placements on a triangular board. Garsia and Remmel [2] defined two statistics on rook placements giving q-analogues of

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the Stirling numbers of the second kind which inspired the work of Wachs and White. And set partitions avoiding certain patterns have received special attention. To illustrate, those avoiding the partition 123 are just matchings or, equivalently, involutions. An interesting statistic on matchings viewed in the setting of rook theory was given by Haglund and Remmel [5].

Let us start by providing the necessary definitions and setting notation. A *set partition* of a set S is a collection σ of nonempty subsets whose disjoint union is S. We write $\sigma = B_1/\dots/B_k \vdash S$ where the subsets B_i are called *blocks*. When no confusion will result, we often drop the curly braces and commas in the B_i. For $[n] = \{1, \dots, n\}$, we use the notation

$$\Pi_n = \{\sigma \ : \ \sigma \vdash [n]\}.$$

To define pattern avoidance in this setting, suppose $\sigma = B_1/\dots/B_k \in \Pi_n$ and $S \subseteq [n]$. Then σ has a corresponding *subpartition* σ' whose blocks are the nonempty intersections $B_i \cap S$. For example, if $\sigma = 14/236/5 \vdash [6]$ and $S = \{2, 4, 6\}$ then $\sigma' = 26/4$. We *standardize* a set partition with integral elements by replacing the smallest element by 1, the next smallest by 2, and so forth. So the standardization of σ' above is 13/2. Given two set partitions σ and π, we say that σ *contains* π *as a pattern* if there is a subpartition of σ which standardizes to π. Otherwise we say that σ *avoids* π. Continuing our example, we have already shown that $\sigma = 14/236/5$ contains 13/2. But σ avoids 123/4 because the only block of σ containing three elements also contains the largest element in σ, so there can be no larger element in a separate block. We let

$$\Pi_n(\pi) = \{\sigma \in \Pi_n \ : \ \sigma \text{ avoids } \pi\}.$$

In order to connect set partitions with the statistics of Wachs and White, we will have to convert them into restricted growth functions. A *restricted growth function* (RGF) is a sequence $w = a_1 \dots a_n$ of positive integers subject to the restrictions

1. $a_1 = 1$, and
2. for $i \geq 2$ we have
$$a_i \leq 1 + \max\{a_1, \dots, a_{i-1}\}. \qquad (1)$$

The number of elements in w is called its *length* and we let

$$R_n = \{w \ : \ w \text{ is an RGF of length } n\}.$$

There is a simple bijection $\Pi_n \to R_n$. We say $\sigma = B_1/\dots/B_k \in \Pi_n$ is in *standard form* if $\min B_1 < \cdots < \min B_k$. Note that this forces $\min B_1 = 1$. We henceforth assume all partitions in Π_n are written in standard form. Associate with σ the word $w(\sigma) = a_1 \dots a_n$ where

$$a_i = j \text{ if and only if } i \in B_j.$$

Using the example from the previous paragraph $w(\sigma) = 122132$. It is easy to see that $w(\sigma)$ is a restricted growth function and that the map $\sigma \mapsto w(\sigma)$ is the desired bijection. It will be useful to have a notation for the RGFs of partitions avoiding a given pattern π, namely

$$R_n(\pi) = \{w(\sigma) \ : \ \sigma \in \Pi_n(\pi)\}.$$

One can express the notion of partition pattern avoidance directly in the language of restricted growth functions. The *canonization* of a sequence $v = b_1 \dots b_k$ of integers is obtained by replacing all copies of the first element of v by 1, all copies of the second different element to appear in v by 2, and so on. For example $v = 55533522312$ canonizes to 11122133243. It follows easily from the definition that the canonization of v is always an RGF. The next result, while not hard to prove, will not be used in the sequel and so we leave its demonstration to the reader.

Proposition 1.1. *Partition σ contains the pattern π if and only if $w(\sigma)$ contains some subsequence which canonizes to $w(\pi)$.* □

Sagan [7] described the set partitions in $\Pi_n(\pi)$ for each $\pi \in \Pi_3$. Although it is not difficult to translate his work into the language of restricted growth functions, we include the proof of the following result for completeness and since it will be used many times subsequently. Define the *initial run* of an RGF w to be the longest prefix of the form $12\dots m$. Also, we will use the notation a^l to indicate a string of l consecutive copies of the letter a in a word. Finally, say that w is *layered* if $w = 1^{n_1}2^{n_2}\dots m^{n_m}$ for positive integers $n_1, n_2, \dots, n_m$.

Theorem 1.2 ([7]). *We have the following characterizations.*

1. $R_n(1/2/3) = \{w \in R_n \ : \ w$ *consists of only* 1*s and* 2*s*$\}$.
2. $R_n(1/23) = \left\{w \in R_n \ : \ w\right.$ *is obtained by inserting a single* 1 *into a word of the form* $1^l23\dots m$ *for some* $l \geq 0$ *and* $m \geq 1 \left.\right\}$.
3. $R_n(13/2) = \{w \in R_n \ : \ w$ *is layered*$\}$.
4. $R_n(12/3) = \{w \in R_n \ : \ w$ *has initial run* $1\dots m$ *and* $a_{m+1} = \cdots = a_n \leq m\}$.
5. $R_n(123) = \{w \in R_n \ : \ w$ *has no element repeated more than twice*$\}$.