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Note

Cycles through an arc in regular 3-partite tournaments[☆]

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ABSTRACT

A c-partite tournament is an orientation of a complete c-partite graph. In 2006, Volkmann conjectured that every arc of a regular 3-partite tournament D is contained in an m-, $(m+1)$- or $(m+2)$-cycle for each $m \in \{3, 4, \ldots, |V(D)|-2\}$, and he also proved this conjecture for $m = 3, 4, 5$. In 2012, Xu et al. proved that every arc of a regular 3-partite tournament is contained in a 5- or 6-cycle, and in the same paper, the authors also posed the following conjecture:

Conjecture 1. If D is an r-regular 3-partite tournament with $r \geq 2$, then every arc of D is contained in a $3k$- or $(3k+1)$-cycle for $k = 1, 2, \ldots, r-1$.

It is known that Conjecture 1 is true for $k = 1$. In this paper, we prove Conjecture 1 for $k = 2$, which implies that Volkmann's conjecture for $m = 6$ is correct.

1. Terminology and introduction

This paper will generally follow the notation and terminology defined in [1], and we assume that all digraphs are finite without loops or multiple arcs. Let D be a digraph. If xy is an arc of D, then we write $x \to y$ and say x *dominates* y. If X and Y are two disjoint vertex subsets of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$, otherwise denoted by $X \nrightarrow Y$. If D' is a vertex set or a subdigraph of D, then we define $N^+_{D'}(x)$ as the set of vertices of D' which are dominated by x and $N^-_{D'}(x)$ as the set of vertices of D' which dominate x. The numbers $d^+_{D'}(x) = |N^+_{D'}(x)|$ and $d^-_{D'}(x) = |N^-_{D'}(x)|$ are called the *out-degree* and *in-degree* of x in D', respectively.

A c-partite tournament is an orientation of a complete c-partite graph, and an m-cycle is a directed cycle of length m. A digraph D is regular, if $d^+(x) = d^-(x) = d^+(y) = d^-(y)$ for all $x, y \in V(D)$.

Multipartite tournaments have been got extensive attention since they were introduced by Moon in 1968. However, many statements on cycles in c-partite tournaments are only valid for $c \geq 4$. At present, the results on 3-partite tournaments are still very few.

For cycles through an arc in a regular 3-partite tournament, Volkmann [2] presented the following conjecture.

Conjecture 1.1 ([2]). *If D is a regular 3-partite tournament, then every arc of D is contained in an m-, $(m+1)$- or $(m+2)$-cycle for each $m \in \{3, 4, \ldots, |V(D)|-2\}$.*

In the same paper, he proved this conjecture for $m = 3, 4, 5$. In 2012, Xu et al. generalized Volkmann's results and proved that every arc of a regular 3-partite tournament is contained in a 5- or 6-cycle, which also implies that the following Conjecture 1.2 is correct for $k = 2$.

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Conjecture 1.2 ([4]). *If D is an r-regular 3-partite tournament with $r \geq 2$, then every arc of D is contained in a $(3k-1)$- or $3k$-cycle for $k = 2, 3, \ldots, r$.*

In the same paper, Xu et al. also posed the following Conjecture 1.3.

Conjecture 1.3 ([4]). *If D is an r-regular 3-partite tournament with $r \geq 2$, then every arc of D is contained in a $3k$- or $(3k+1)$-cycle for $k = 1, 2, \ldots, r-1$.*

Volkmann [2] proved that every arc of a regular 3-partite tournament D is contained in a 3- or 4-cycle, which implies that Conjecture 1.3 is true for $k = 1$. In this paper, we prove Conjecture 1.3 for $k = 2$ and Conjecture 1.1 for $m = 6$.

More information on cycles of 3-partite tournaments are contained in the paper by Volkmann [3].

2. Main result

The following two lemmas are important for the proof of main result.

Lemma 2.1 ([4]). *If D is an r-regular 3-partite tournament with partite sets V_1, V_2, V_3 and v is a vertex of D, then $|V_1| = |V_2| = |V_3| = r$ and $d^+(v) = d^-(v) = r$.*

Lemma 2.2 ([4]). *If D is an r-regular 3-partite tournament with partite sets U, V, W and u is a vertex of U, then $d^+_V(u) = d^-_W(u)$ and $d^-_V(u) = d^+_W(u)$.*

Theorem 2.1. *If D is an r-regular 3-partite tournament with $r \geq 3$, then every arc of D is contained in a 6- or 7-cycle.*

Proof. Let V_1, V_2, V_3 be the partite sets of D, and let ab be an arbitrary arc of D. Without loss of generality, we suppose that $a \in V_1$ and $b \in V_2$. We distinguish the following two cases.

Case 1 $V_3 \to a \to V_2$.

By Lemma 2.2, we have $d^+_{V_3}(x) \geq 1$ and $d^-_{V_2}(y) \geq 1$ for any $x \in V_2, y \in V_3$. So there are at least r arcs from V_2 to V_3. If there are exactly r arcs from V_2 to V_3, we array the vertices of V_2 into $b_1, b_2, \ldots, b_r$ with $b_1 = b$ and the vertices of V_3 into $c_1, c_2, \ldots, c_r$ such that $b_i \to c_i$ and $V_3 - \{c_i\} \to b_i, i = 1, 2, \ldots, r$. Obviously, $abc_1b_2c_2b_3c_3a$ is a 7-cycle through ab. Therefore, we assume that there are more than r arcs from V_2 to V_3.

If $V_2 \not\Rightarrow N^+_{V_3}(b)$, then there exists a vertex $x \in N^+_{V_3}(b)$ and a vertex $y \in V_2$ such that $x \to y$. By Lemma 2.2, there is an arc yz where $z \in V_1 - \{a\}$. When $V_3 - \{x\} \not\Rightarrow z$, there is a vertex u in $V_3 - \{x\}$ such that $z \to u$. So $abxyzua$ is a 6-cycle through ab. When $V_3 - \{x\} \to z$, we have $z \to V_2 - \{y\}$ (otherwise $|N^-(z)| > r$, a contradiction). Let $u \in V_2 - \{b, y\}$ be arbitrary. Since $\{a, z\} \to u$ and by Lemma 2.2, there is an arc uv where $v \in V_3 - \{x\}$ and $abxyzuva$ is a 7-cycle through ab.

We assume that $V_2 \to N^+_{V_3}(b)$. If $N^+_{V_3}(b) = V_3$, then we have $V_1 \to V_2 \to V_3 \to V_1$. So $abxyzua$ is a 6-cycle through ab where $y \in V_1 - \{a\}, z \in V_2 - \{b\}, \{x, u\} \in V_3$ and $x \neq u$. If $N^+_{V_3}(b) \neq V_3$, let $x \in N^+_{V_3}(b)$ be arbitrary. Then we have that $V_2 \to x \to V_1$. By Lemma 2.2, there exists a vertex $y \in V_1 - \{a\}$ and a vertex $z \in V_2$ such that $y \to z$. We can choose $z \neq b$ (otherwise, $V_1 \to b$ and thus $b \to V_3$, which is a contradiction with $N^+_{V_3}(b) \neq V_3$). Since $\{a, y\} \to z$ and by Lemma 2.2, there is an arc zu where $u \in V_3 - \{x\}$ and $abxyzua$ is a 6-cycle through ab.

Case 2 $V_3 \not\Rightarrow a \not\Rightarrow V_2$.

Obviously, the partite set V_2 can be divided into two nonempty parts V_2^+, V_2^- such that $V_2^- \to a \to V_2^+$. Similarly, the partite set V_3 can be divided into two nonempty parts V_3^+, V_3^- such that $V_3^- \to a \to V_3^+$. Let $V' = V_2^+ \cup V_3^+$ and $V'' = V_2^- \cup V_3^-$. Observe that $N^+(a) = V', N^-(a) = V''$ and $|V'| = |V''| = r$. By Lemma 2.1, we also have $|V_2^+| = |V_3^-|$ and $|V_2^-| = |V_3^+|$.

Claim 1 Let $y \in V_1$ be arbitrary. Then $d^+_{V'}(y) = d^-_{V''}(y)$ and $d^-_{V'}(y) = d^+_{V''}(y)$.

Proof By Lemma 2.1, we have $d^+_{V'}(y) + d^-_{V'}(y) = |V'| = r$, $d^+_{V''}(y) + d^-_{V''}(y) = |V''| = r$, $d^+_{V'}(y) + d^+_{V''}(y) = d^+(y) = r$ and $d^-_{V'}(y) + d^-_{V''}(y) = d^-(y) = r$. So $d^+_{V'}(y) = d^-_{V''}(y)$ and $d^-_{V'}(y) = d^+_{V''}(y)$. □

Subcase 2.1 $V_3^+ \not\Rightarrow b$.

Since $V_3^+ \not\Rightarrow b$, there is a vertex $x \in V_3^+$ such that $b \to x$.

Subcase 2.1.1 $|V_2^+| = r - 1$.

In this case, we have $|V_2^+| = |V_3^-| = r - 1$, $|V_2^-| = |V_3^+| = 1$ and $V_3^+ = \{x\}$. If $V_2^+ \to x$, then $N^-(x) = V_2^+ \cup \{a\}$ and $x \to (V_1 - \{a\}) \cup V_2^-$. Let $y \in V_1 - \{a\}$ and $V_2^- = \{z\}$. If $y \to z$, Lemma 2.2 implies that there exists a vertex $u \in V_3^-$ such that $z \to u$ and $abxyzua$ is a 6-cycle through ab. By Lemma 2.2, if $z \to y$, then there is a vertex $v \in V_3^-$ such that $y \to v$ and $abxzyva$ is a 6-cycle through ab.