Discrete Mathematics 339 (2016) 21–32

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Bounds and constructions for optimal $(n, \{3, 5\}, \Lambda_a, 1, Q)$-OOCs

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ARTICLE INFO

Article history:
Received 27 November 2014
Received in revised form 15 July 2015
Accepted 19 July 2015
Available online 5 August 2015

Keywords:
Cyclic packing
Optimal
Regular
Skew starter
Variable-weight optical orthogonal codes

ABSTRACT

Let $W = \{w_1, w_2, \ldots, w_r\}$ be an ordering of a set of r integers greater than 1, $\Lambda_a = (\lambda_a^{(1)}, \lambda_a^{(2)}, \ldots, \lambda_a^{(r)})$ be an r-tuple of positive integers, λ_c be a positive integer, and $Q = (q_1, q_2, \ldots, q_r)$ be an r-tuple of positive rational numbers whose sum is 1. In 1996, Yang introduced variable-weight optical orthogonal code ($(n, W, \Lambda_a, \lambda_c, Q)$-OOC) for multimedia optical CDMA systems with multiple quality of service (QoS) requirements. Some work had been done on the constructions of optimal $(n, \{3, 4\}, \Lambda_a, 1, Q)$-OOCs with unequal auto- and cross-correlation constraints. In this paper, we focus our main attention on $(n, \{3, 5\}, \Lambda_a, 1, Q)$-OOCs, where $\Lambda_a \in \{(1, 2), (2, 1), (2, 2)\}$. Tight upper bounds on the maximum code size of an $(n, \{3, 5\}, \Lambda_a, 1, Q)$-OOC are obtained, and infinite classes of optimal balanced $(n, \{3, 5\}, \Lambda_a, 1)$-OOCs are constructed.

1. Introduction

Let $n,\ k,\ \lambda_a,\ \lambda_c$ be positive integers. Optical orthogonal codes, $(n, k, \lambda_a, \lambda_c)$-OOCs, were introduced by Salehi, as signature sequences to facilitate multiple access in optical fiber networks [23,24]. They have important applications [16]. Many existing works on OOCs have assumed that $\lambda_a = \lambda_c = 1$, i.e. $(n, k, 1)$-OOCs, see [1,4,11,14,18,19,28] for some of the examples. Recently, $(n, k, 2, 1)$-OOCs had been deeply investigated for $3 \le k \le 5$ [3,8,12,22,25].

To meet multiple QoS requirements, Yang introduced multimedia optical CDMA communication system employing variable-weight OOCs in 1996 [27]. In the system, codes with different weights will have different bit error rates (BER). The codewords of low code weight can be assigned to the low-QoS applications and high code weight codewords can be assigned to high-QoS requirement applications. Hence, the multi-weight property of the OOCs enables the system to meet multiple QoS requirements. Similar to [22], throughout this paper, let $\binom{Z_n}{k}$ be the set of all k-subsets of Z_n, the residue ring of integers modulo n. The following definition of a variable-weight OOC is from [20].

Let $W = \{w_1, \ldots, w_r\}$ be an ordering of a set of r integers greater than 1, $\Lambda_a = (\lambda_a^{(1)}, \ldots, \lambda_a^{(r)})$ an r-tuple (*auto-correlation sequence*) of positive integers, λ_c a positive integer (*cross-correlation parameter*), and $Q = (q_1, \ldots, q_r)$ an r-tuple (*weight distribution sequence*) of positive rational numbers whose sum is 1.

Definition 1. An $(n, W, \Lambda_a, \lambda_c, Q)$-OOC is a set $\mathscr{C}$ of subsets (*codeword-sets*) of Z_n with sizes (*weights*) from W satisfying the following properties:

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http://dx.doi.org/10.1016/j.disc.2015.07.006

weight distribution property: the ratio of codeword-sets of $\mathscr{C}$ of weight w_i is q_i:

$$\left|\mathscr{C} \cap \binom{Z_n}{w_i}\right| = q_i|\mathscr{C}| \quad \text{for } 1 \le i \le r;$$

auto-correlation property: any two distinct translates of a codeword-set of weight w_i share at most $\lambda_a^{(i)}$ elements:

$$|C \cap (C+t)| \le \lambda_a^{(i)} \quad \forall\, C \in \mathscr{C} \cap \binom{Z_n}{w_i},\ \forall\, t \in Z_n \setminus \{0\}; \tag{1}$$

cross-correlation property: any two translates of two distinct codeword-sets share at most λ_c elements:

$$|C \cap (C'+t)| \le \lambda_c \quad \forall\, \{C, C'\} \in \binom{\mathscr{C}}{2},\ \forall\, t \in Z_n. \tag{2}$$

If $\lambda_a^{(i)} = \lambda_a$ for every i, one simply says that $\mathscr{C}$ is an $(n, W, \lambda_a, \lambda_c, Q)$-OOC. Also, speaking of an (n, W, λ, Q)-OOC one means an $(n, W, \lambda_a, \lambda_c, Q)$-OOC where $\lambda_a = \lambda_c = \lambda$. We say that Q is *normalized* if it is written in the form $Q = (\frac{a_1}{b}, \ldots, \frac{a_r}{b})$ with $\gcd(a_1, \ldots, a_r) = 1$. Speaking of a *balanced* $(n, W, \Lambda_a, \lambda_c)$-OOC we mean an $(n, W, \Lambda_a, \lambda_c, Q)$-OOC with $Q = (\frac{1}{r}, \frac{1}{r}, \ldots, \frac{1}{r})$, namely an OOC in which the number of codeword-sets of a given weight is a constant. The term variable-weight optical orthogonal code, or variable-weight OOC, is also used if there is no need to list the parameters. The code $\mathscr{C}$ is said to be optimal if its size reaches a suitable upper bound deduced from general theoretical considerations.

Example 1. A $(33, \{3, 5\}, (1, 1), 1, (2/3, 1/3))$-OOC $\mathscr{C}_1$.

$\mathscr{C}_1 = \{\{0, 5, 18\}, \{0, 8, 19\}, \{0, 1, 3, 7, 24\}\}$.

Some work had been done on the existences of $(n, W, 1, Q)$-OOCs. The interested reader may refer to [13,26,27] and the references therein. Some results on the code size of an $(n, \{3, 4\}, \Lambda_a, 1, Q)$-OOC and the constructions of optimal $(n, \{3, 4\}, \Lambda_a, 1, Q)$-OOCs are presented in [20,29]. In this paper, the upper bounds on the code size of $(n, \{3, 5\}, \Lambda_a, 1, Q)$-OOCs are obtained for $\Lambda_a \in \{(1, 2), (2, 1), (2, 2)\}$, and infinite classes of optimal balanced $(n, \{3, 5\}, \Lambda_a, 1)$-OOCs are constructed.

2. Preliminaries

We will use the terminology in [22] and [8]. For a given subset C of Z_n, let us denote by ΔC its *list of differences*, which is the multiset of all differences $x - y$ with (x, y) an ordered pair of distinct elements of C. The underlying set of ΔC, i.e. the set of all distinct differences appearing in ΔC, will be called the *support* of ΔC and we denote it by $supp(\Delta C)$. More generally, the list of differences of a set $\mathscr{C}$ of subsets of Z_n is the multiset $\Delta\mathscr{C} = \bigcup_{C \in \mathscr{C}} \Delta C$. Also, the support of $\Delta\mathscr{C}$ is the underlying set of $\bigcup_{C \in \mathscr{C}} \Delta C$. Saying that $\mathscr{C}$ is *difference-disjoint* we mean that the lists of differences of every two distinct members of $\mathscr{C}$ are disjoint. Define $\lambda(C) = \max\{m_d(\Delta C) : d \in \Delta C\}$, where $m_d(\Delta C)$ denotes the multiplicity of d in ΔC. Then we have $\lambda(C) = \max\{|C \bigcap (C + i)| : i \in Z_n \setminus \{0\}\}$. From this correspondence, one can see the following.

Lemma 2.1. *Every $(n, W, \Lambda_a, 1, Q)$-OOC can be viewed as a family $\mathscr{C}$ of subsets of Z_n satisfying the following conditions:*
(i) *The ratio of codeword-sets of weight w_i is q_i, $1 \le i \le r$;*
(ii) *For every $C \in \mathscr{C}$ with weight w_i, we have $\lambda(C) \le \lambda_a^{(i)}$;*
(iii) *$\mathscr{C}$ is difference-disjoint.*

The difference leave of an $(n, W, \Lambda_a, 1, Q)$-OOC $\mathscr{C}$, denoted by $\mathrm{DL}(\mathscr{C})$, is the set of all elements of Z_n which are not covered by $\Delta\mathscr{C}$. We say that $\mathscr{C}$ is m-regular if $\mathrm{DL}(\mathscr{C})$ is the subgroup of Z_n of order m.

Lemma 2.2 ([20,22]). *For $C \in \binom{Z_n}{3}$,*

$$|supp(\Delta C)| = \begin{cases} 2 & \text{if } C = \left(\frac{n}{3}\right) Z_n; \\ 3 & \text{if } C \subset \left(\frac{n}{4}\right) Z_n; \\ 4 & \text{if } C = \{0, a, 2a\} \text{ and } a \notin \{0, \pm n/3, \pm n/4\}; \\ 5 & \text{if } C = \{0, a, n/2\} \text{ and } a \notin \{0, \pm n/4\}, \end{cases}$$

$$\lambda(C) = \begin{cases} 3 & \text{if } |supp(\Delta C)| = 2; \\ 2 & \text{if } |supp(\Delta C)| = 3, 4, \text{ or } 5; \\ 1 & \text{if } |supp(\Delta C)| = 6, \end{cases}$$

and