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An improved upper bound on the linear 2-arboricity of planar graphs

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ABSTRACT

The linear 2-arboricity $la_2(G)$ of a graph G is the least integer k such that G can be partitioned into k edge-disjoint forests, whose component trees are paths of length at most 2. In this paper, we prove that if G is a planar graph, then $la_2(G) \leq \lceil (\Delta(G) + 1)/2 \rceil + 6$. This improves a result in Lih et al. (2003), which says that every planar graph G satisfies $la_2(G) \leq \lceil (\Delta(G) + 1)/2 \rceil + 12$.

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1. Introduction

All graphs considered in this paper are finite simple graphs. For a graph G, we use $V(G)$, $E(G)$, $\Delta(G)$, and $\delta(G)$, to denote, respectively, its vertex set, edge set, maximum degree, and minimum degree. An *edge-partition* of a graph G is a decomposition of G into subgraphs $G_1, G_2, \dots, G_m$ such that $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_m)$ and $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$. A *linear k-forest* is a graph whose components are paths of length at most k. The *linear k-arboricity* of G, denoted by $la_k(G)$, is the least integer m such that G can be edge-partitioned into m linear k-forests. Clearly, $la_k(G) \geq la_{k+1}(G)$ for any $k \geq 1$. For extremities, $la_1(G)$ is the chromatic index $\chi'(G)$ of G; $la_\infty(G)$ corresponds to the linear arboricity $la(G)$ of G.

The linear k-arboricity of a graph was first introduced by Habib and Péroche [9]. For any graph G on n vertices, they put forward the following conjecture:

$$la_k(G) \leq \begin{cases} \left\lceil \dfrac{n\Delta(G)+1}{2\lfloor \frac{kn}{k+1} \rfloor} \right\rceil & \text{if } \Delta(G) \neq n-1; \\ \left\lceil \dfrac{n\Delta(G)}{2\lfloor \frac{kn}{k+1} \rfloor} \right\rceil & \text{if } \Delta(G) = n-1. \end{cases}$$

This notion was further studied by Bermond et al. [2], Jackson and Wormald [10], Aldred and Wormald [1], Chen and Huang [6], Chang et al. [5], and Thomassen [15]. Moreover, Chang [4] discussed the algorithmic aspects of the linear k-arboricity.

In recent years, a number of interesting results about the linear 2-arboricity of graphs have been obtained. First, we note that when $k = 2$, Conjecture 1 is written as follows:

$$la_2(G) \leq \begin{cases} \left\lceil \dfrac{n\Delta(G)+1}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil & \text{if } \Delta(G) \neq n-1; \\ \left\lceil \dfrac{n\Delta(G)}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil & \text{if } \Delta(G) = n-1. \end{cases}$$

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In [7,8], the linear 2-arboricities of some special graphs such as cycles, trees, complete graphs, and complete bipartite graphs were determined. Suppose that G is a planar graph with the girth g. Set $\eta(G) = \lceil(\Delta(G)+1)/2\rceil$. Lih, Tong and Wang [11] proved that (a) $\mathrm{la}_2(G) \le \eta(G)+12$; (b) $\mathrm{la}_2(G) \le \eta(G)+6$ if $g \ge 4$; (c) $\mathrm{la}_2(G) \le \eta(G)+2$ if $g \ge 5$; (d) $\mathrm{la}_2(G) \le \eta(G)+1$ if $g \ge 7$. Qian and Wang [14] proved that $\mathrm{la}_2(G) \le \eta(G)+3$ if G contains no 4-cycles. Ma et al. [13] proved that $\mathrm{la}_2(G) \le \eta(G)+6$ if G contains no 5-cycles or 6-cycles. For an outerplanar graph G, Lih et al. [12] proved that $\mathrm{la}_2(G) \le \eta(G)+1$ and this upper bound is tight.

In this paper, we show that for any planar graph G, $\mathrm{la}_2(G) \le \eta(G)+6$, which improves a result in [11].

To obtain our main result, we need to introduce some notions. A *plane* graph is a particular drawing of a planar graph on the Euclidean plane. For a plane graph G, let $F(G)$ denote the face set of G. For $f \in F(G)$, we use $b(f)$ to denote the closed boundary walk of f and write $f = [u_1u_2\cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices on the boundary walk in the clockwise order, with repeated occurrences of vertices allowed. A vertex of degree k (at most k, at least k) is called a k-*vertex* (k^--vertex, k^+-vertex). Similarly, we can define k-face, k^--face, and k^+-face.

2. Edge-partition

In this section, we establish an edge-partition theorem on planar graphs, which will play an important role in the proof of our main result. We first give the following useful fact:

Lemma 1 ([3]). *Every planar graph G with $\delta(G) \ge 2$ contains one of the following configurations:*
(1) *An edge xy such that $d_G(x)+d_G(y) \le 15$;*
(2) *A cycle $v_0v_1\cdots v_{2s-1}v_0$ such that $d_G(v_0) = d_G(v_2) = \cdots = d_G(v_{2s-2}) = 2$.*

Theorem 2. *Every planar graph G has an edge-partition into two forests F_1, F_2 and a subgraph H such that, for every $v \in V(G)$, $d_H(v) \le 11$ and $d_{F_i}(v) \le \max\{2, \lceil\frac{d_G(v)-11}{2}\rceil\}$ for $i = 1, 2$.*

Proof. We prove the theorem by induction on the edge number $|E(G)|$. If $|E(G)| \le 11$, then the result holds trivially. Let G be a planar graph with $|E(G)| \ge 12$. If $\Delta(G) \le 11$, it suffices to take $H = G$ and $F_1 = F_2 = \emptyset$. Assume that $\Delta(G) \ge 12$. If G' is a proper spanning subgraph of G, then G' has an edge-partition into two forests F_1', F_2' and a subgraph H' such that, for every $t \in V(G')$, $d_{H'}(t) \le 11$ and $d_{F_i'}(t) \le \max\{2, \lceil\frac{d_{G'}(t)-11}{2}\rceil\}$ for $i = 1, 2$, by the induction hypothesis. We are going to choose appropriate subgraphs G' so that we can extend $F_1' \cup F_2' \cup H'$ to an edge-partition $F_1 \cup F_2 \cup H$ of G satisfying the theorem.

In the following, for any vertex t, we simply write:

$$\beta(t) = \max\left\{2, \left\lceil\frac{d_G(t)-11}{2}\right\rceil\right\},$$

and

$$\beta'(t) = \max\left\{2, \left\lceil\frac{d_{G'}(t)-11}{2}\right\rceil\right\}.$$

Since G' is a spanning subgraph of G, for any $t \in V(G') = V(G)$, $d_{G'}(t) \le d_G(t)$, and hence $\beta'(t) \le \beta(t)$.

If $\delta(G) = 1$, let $uv \in E(G)$ with $d_G(u) = 1$. Consider the graph $G' = G - uv$. By the induction hypothesis, G' admits an edge-partition into two forests F_1', F_2' and a subgraph H' such that, for every $t \in V(G')$, $d_{H'}(t) \le 11$ and $d_{F_i'}(t) \le \beta'(t)$ for $i = 1, 2$.

If $d_{H'}(v) \le 10$, then we define $H = H' + uv$ and $F_i = F_i'$ for $i = 1, 2$. It is easy to inspect that $F_1 \cup F_2 \cup H$ is an edge-partition of G satisfying the theorem. Otherwise, $d_{H'}(v) = 11$. We suppose that $d_{F_1'}(v) \le d_{F_2'}(v)$. Since $d_{G'}(v) = d_{F_1'}(v)+d_{F_2'}(v)+d_{H'}(v) = d_{F_1'}(v)+d_{F_2'}(v)+11$ and $d_{G'}(v) = d_G(v)-1$, we have $d_{F_1'}(v) \le \lfloor(d_G(v)-12)/2\rfloor$. Let $F_1 = F_1'+uv$, $F_2 = F_2'$, and $H = H'$. Thus, $d_{F_2}(t) = d_{F_2'}(t)$ and $d_H(t) = d_{H'}(t)$ for all $t \in V(G)$. Moreover, $d_{F_1}(u) = 1 < 2 = \beta(u)$, $d_{F_1}(v) = 1 + d_{F_1'}(v) \le 1 + \lfloor(d_G(v)-12)/2\rfloor = \lceil(d_G(v)-11)/2\rceil \le \beta(v)$, and $d_{F_1}(t) = d_{F_1'}(t) \le \beta'(t) \le \beta(t)$ for all $t \in V(G)\setminus\{u, v\}$.

If $\delta(G) \ge 2$, by Lemma 1, we consider two cases as follows.

Case 1. There is an edge $xy \in E(G)$ such that $d_G(x)+d_G(y) \le 15$.

Let $G' = G - xy$. By the induction hypothesis, G' admits an edge-partition into two forests F_1', F_2' and a subgraph H' such that, for every $t \in V(G')$, $d_{H'}(t) \le 11$ and $d_{F_i'}(t) \le \beta'(t)$ for $i = 1, 2$. Without loss of generality, assume that $d_{H'}(x) \le d_{H'}(y)$. If $d_{H'}(y) \le 10$, let $H = H'+xy$, $F_1 = F_1'$, and $F_2 = F_2'$. It is easy to verify that $F_1 \cup F_2 \cup H$ is an edge-partition of G satisfying the theorem.

Assume that $d_{H'}(y) = 11$. Since $\delta(G) \ge 2$ and $d_G(x)+d_G(y) \le 15$, it is immediate to derive that $d_{G'}(x)+d_{G'}(y) \le 13$. Thus, $1 \le d_{G'}(x) \le 2$ and $d_{F_1'}(y)+d_{F_2'}(y)+d_{G'}(x) \le 2$. This further implies that $d_{F_1'}(y)+d_{F_2'}(y) \le 1$. Without loss of generality, we suppose that $d_{F_1'}(x) \le d_{F_2'}(x)$. Thus, $d_{F_1'}(x) \le 1$. Let $F_1 = F_1'+xy$, $F_2 = F_2'$, and $H = H'$. It is easy to see that $d_{F_1}(x) = d_{F_1'}(x)+1 \le 1+1 = 2 \le \beta(x)$, $d_{F_1}(y) = d_{F_1'}(y)+1 \le 1+1 = 2 \le \beta(y)$, $d_{F_2}(x) = d_{F_2'}(x)$, $d_{F_2}(y) = d_{F_2'}(y)$, $d_H(x) = d_{H'}(x) \le 2$, and $d_H(y) = d_{H'}(y) = 11$.

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