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Monochromatic bounded degree subgraph partitions

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ABSTRACT

Let $\mathcal{F} = \{F_1, F_2, \ldots\}$ be a sequence of graphs such that F_n is a graph on n vertices with maximum degree at most Δ. We show that there exists an absolute constant C such that the vertices of any 2-edge-colored complete graph can be partitioned into at most $2^{C\Delta \log \Delta}$ vertex disjoint monochromatic copies of graphs from $\mathcal{F}$. If each F_n is bipartite, then we can improve this bound to $2^{C\Delta}$; this result is optimal up to the constant C.

1. Introduction

Let K_n be a complete graph on n vertices whose edges are colored with r colors ($r \geq 1$). How many monochromatic cycles (single vertices and edges are considered to be cycles) are needed to partition the vertex set of K_n? This question received much attention in the last few years. Let $p(r)$ denote the minimum number of monochromatic cycles needed to partition the vertex set of any r-colored K_n. It is not obvious that $p(r)$ is a well-defined function. That is, it is not obvious that there always is a partition whose cardinality is independent of n. However, in [18] Erdős, Gyárfás, and Pyber proved that there exists a constant C such that $p(r) \leq Cr^2 \log r$ (throughout this paper log denotes the natural logarithm). Furthermore, in [18] (see also [26]), the authors conjectured that $p(r) = r$.

The special case $r = 2$ of this conjecture was asked earlier by Lehel and for $n \geq n_0$ was first proved by Łuczak, Rödl, and Szemerédi [41]. Allen improved on the value of n_0 [1] and recently Bessy and Thomassé [3] proved the original conjecture for $r = 2$. For general r the current best bound is due to Gyárfás, Ruszinkó, Sárközy, and Szemerédi [27] who proved that for $n \geq n_0(r)$ we have $p(r) \leq 100r \log r$. For $r = 3$ an approximate version of the conjecture was proved in [28] but, surprisingly, Pokrovskiy [43] found a counterexample to the conjecture. However, in the counterexample, all but one vertex can be covered by r vertex disjoint monochromatic cycles. Thus, a slightly weaker version of the conjecture still can be true, say that, apart from a constant number of vertices, the vertex set can be covered by r vertex disjoint monochromatic cycles.

Let us also note that the above problem was generalized in various directions; for hypergraphs (see [29] and [48]), for complete bipartite graphs (see [18] and [31]), for graphs which are not necessarily complete (see [2] and [47]), and for partitions by monochromatic connected k-regular graphs (see [50] and [51]).

Another area that attracted a lot of interest is the study of Ramsey numbers for bounded degree graphs. For a graph G, the *Ramsey number* $R(G)$ is the smallest positive integer N such that if the edges of a complete graph K_N are partitioned into two color classes then one color class has a subgraph isomorphic to G. The existence of such a positive integer is guaranteed by Ramsey's classical result [45]. Determining $R(G)$ even for very special graphs is notoriously hard (see e.g. [25] or [44]).

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In 1975, Burr and Erdős [6] raised the question whether that every graph G with n vertices and maximum degree Δ has a linear Ramsey number, so $R(G) \leq C(\Delta)n$, for some constant $C(\Delta)$ depending only on Δ. This was proved by Chvátal, Rödl, Szemerédi and Trotter [9] in one of the earliest applications of Szemerédi's celebrated Regularity Lemma [52]. Since the proof uses the Regularity Lemma, the bound on $C(\Delta)$ is quite weak; it is of tower type in Δ. This was improved by Eaton [17], who proved, using a variant of the Regularity Lemma, that the function $C(\Delta)$ can be taken to be of the form $2^{2^{O(\Delta)}}$.

Soon after, Graham, Rödl, and Ruciński [24] improved this further to $C(\Delta) \leq 2^{O(\Delta \log^2 \Delta)}$ and for bipartite graphs to $C(\Delta) \leq 2^{O(\Delta \log \Delta)}$. They also proved that there are bipartite graphs with n vertices and maximum degree Δ for which the Ramsey number is at least $2^{\Omega(\Delta)}n$. Recently, Conlon [10] and, independently, Fox and Sudakov [23] have shown how to remove the $\log \Delta$ factor in the exponent, achieving an essentially best possible bound of $R(G) \leq 2^{O(\Delta)}n$ in the bipartite case. For the non-bipartite graph case, the current best bound is due to Conlon, Fox, and Sudakov [13] $C(\Delta) \leq 2^{O(\Delta \log \Delta)}$. Similar results have been proven for hypergraphs: [14,15,42] use the Hypergraph Regularity Lemma and [12] improves the bounds by avoiding the Regularity Lemma.

It is a natural question (initiated by András Gyárfás) to combine the above two problems and ask how many monochromatic members from a bounded degree graph family are needed to partition the vertex set of a 2-edge-colored K_N. In this paper we study this problem. Given a sequence $\mathcal{F} = \{F_1, F_2, \ldots\}$ of graphs, we say it is *Δ-bounded* if each F_n is a graph on n vertices with maximum degree at most Δ. In general we say that $\mathcal{F}$ has some graph property if every graph of $\mathcal{F}$ has that property (e.g. $\mathcal{F}$ is bipartite if F_n is bipartite for every n).

We prove the following result on partitions by monochromatic members of $\mathcal{F}$.

Theorem 1. *There exists an absolute constant C such that, for every $\Delta \geq 2$ and every Δ-bounded graph sequence $\mathcal{F}$, every 2-edge-colored complete graph can be partitioned into at most $2^{C\Delta \log \Delta}$ vertex disjoint monochromatic copies of graphs from $\mathcal{F}$.*

Thus, perhaps surprisingly, we have the same phenomenon as for cycles; we can partition into monochromatic graphs from $\mathcal{F}$ such that their average size is roughly the same as the single largest monochromatic graph we can find. In the case of a bipartite $\mathcal{F}$ we can eliminate the $\log \Delta$ factor from the exponent to get the following essentially best possible result.

Theorem 2. *There exists an absolute constant C such that, for every Δ and every bipartite Δ-bounded graph sequence $\mathcal{F}$, every 2-edge-colored complete graph can be partitioned into at most $2^{C\Delta}$ vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$.*

We do not make an effort to optimize the constant C since probably it will be far from optimal anyway. However, in both theorems we must use at least $2^{\Omega(\Delta)}$ parts.

Theorem 3. *There exists an absolute constant c such that, for every Δ, there is a bipartite Δ-bounded graph sequence $\mathcal{F}$ and there is a 2-edge-coloring of K_n so that covering the vertices of K_n using monochromatic copies of graphs from $\mathcal{F}$ requires at least $2^{c\Delta}$ such copies.*

It would be desirable to close the gap between the upper and lower bounds for non-bipartite $\mathcal{F}$ as well, though doing so may require improved bounds for the Ramsey numbers of bounded degree graphs. Furthermore, it would be interesting to extend this problem for more than 2 colors.

Let us also mention one interesting special case of our theorem. The kth *power* of a cycle C is the graph obtained from C by joining every pair of vertices with distance at most k in C. Density questions for powers of cycles have generated a lot of interest; in particular the famous Pósa–Seymour conjecture (see e.g. [7,19–22,34,36,37,40]). Theorem 1 implies the following result on the partition number by monochromatic powers of cycles.

Corollary 1. *There exists an absolute constant C so that for every k every 2-colored complete graph can be partitioned into at most $2^{Ck \log k}$ vertex disjoint monochromatic kth powers of cycles.*

However, we must note that in this case probably the optimal answer is $O(k)$.

2. Notation and tools

For basic graph concepts see the monograph of Bollobás [4].

$V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. Let (A, B, E) denote a bipartite graph $G = (V, E)$, where $V = A \cup B$ and $E \subset A \times B$. A proper r-coloring of G is a coloring of its vertices where no two adjacent vertices receive the same color. For a graph G and a subset U of its vertices, $G|_U$ is the restriction to U of G. Let $N(v)$ denote the set of neighbors of $v \in V$. Hence, $|N(v)| = deg(v) = deg_G(v)$, the degree of v. Let $\delta(G)$ stand for the minimum and $\Delta(G)$ for the maximum degree in G. When A, B are subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of G with one endpoint in A and the other in B. In particular, we write $deg(v, U) = e(\{v\}, U)$ for the number of edges from v to U. For non-empty A and B,

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

is the *density* of the graph between A and B.