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Projective duality of arrangements with quadratic logarithmic vector fields

Ştefan O. Tohǎneanu

Department of Mathematics, University of Idaho, Moscow, ID 83844-1103, USA

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ABSTRACT

In these notes we study hyperplane arrangements having at least one logarithmic derivation of degree two that is not a combination of degree one logarithmic derivations. It is well-known that if a hyperplane arrangement has a linear logarithmic derivation not a constant multiple of the Euler derivation, then the arrangement decomposes as the direct product of smaller arrangements. The next natural step would be to study arrangements with non-trivial quadratic logarithmic derivations. On this regard, we present a computational lemma that leads to a full classification of hyperplane arrangements of rank 3 having such a quadratic logarithmic derivation. These results come as a consequence of looking at the variety of the points dual to the hyperplanes in such special arrangements.

1. Introduction

Let $\mathcal{A}$ be a central essential hyperplane arrangement in V a vector space of dimension k over $\mathbb{K}$ a field of characteristic zero. Let $R = Sym(V^*) = \mathbb{K}[x_1, \ldots, x_k]$ and fix $\ell_i \in R$, $i = 1, \ldots, n$ the linear forms defining the hyperplanes of $\mathcal{A}$. After a change of coordinates, assume that $\ell_i = x_i$, $i = 1, \ldots, k$.

A *logarithmic derivation* (or *logarithmic vector field*) of $\mathcal{A}$ is an element $\theta \in Der(R)$, such that $\theta(\ell_i) \in \langle \ell_i \rangle$, for all $i = 1, \ldots, n$. Picking the standard basis for $Der(R)$, i.e., $\partial_1 := \partial_{x_1}, \ldots, \partial_k := \partial_{x_k}$, if θ is written as

$$\theta = \sum_{i=1}^{k} P_i \partial_i,$$

where $P_i \in R$ are homogeneous polynomials of the same degree, then $\deg(\theta) = \deg(P_i)$. The set of logarithmic derivations forms an R-module, and whenever this module is free one says that the hyperplane arrangement is *free*.

In general, every central hyperplane arrangement has *the Euler derivation*:

$$\theta_E = x_1\partial_1 + \cdots + x_k\partial_k.$$

There exists a one-to-one correspondence between logarithmic derivations not multiples of θ_E and the first syzygies on the *Jacobian ideal of* $\mathcal{A}$, which is the ideal of R generated by the (first order) partial derivatives of the defining polynomial of $\mathcal{A}$.[1] Therefore, we are interested in hyperplane arrangements that have a minimal quadratic syzygy on its Jacobian ideal. Throughout the notes we are going to use both terminologies: "non-trivial quadratic logarithmic derivation" or "minimal quadratic syzygy".

E-mail address: tohaneanu@uidaho.edu.

[1] For more details about this, and in general about the theory of hyperplane arrangements, the first place to look is the landmark book of Orlik and Terao, [2].

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[6] presents interesting constructions of hyperplane arrangements with linear or quadratic syzygies. These are summed up in Proposition 8.5: if $\mathcal{A}$ is an essential arrangement with $e(\mathcal{A}') \leq 2$, for all subarrangements $\mathcal{A}' \subseteq \mathcal{A}$, then the Jacobian ideal of $\mathcal{A}$ has only linear or quadratic syzygies, which are combinatorially constructed. Here $e(\mathcal{A}') = \min_{H\in\mathcal{A}'}(|\mathcal{A}'|-|\mathcal{A}'|_H|)$ is the *excess* of $\mathcal{A}'$. A question comes up immediately: is it true that a free hyperplane arrangement with exponents 1's and 2's is supersolvable? The answer would be yes, if one shows that a free arrangement with exponents 1's and 2's has quadratic Orlik–Terao algebra. Then, by using [1, Theorems 5.1 and 5.11], one obtains supersolvability.

In these notes we do not discuss the freeness of the hyperplane arrangements we study. We are more interested in the geometry of the configuration of points that are dual to the hyperplanes of an arrangement that has a minimal quadratic syzygy on its Jacobian ideal.

At the beginning of the next section we briefly review [2, Proposition 4.29(3)] that characterizes hyperplane arrangements with a linear syzygy, and we look at this result from the projective duality view mentioned already. Next we study hyperplane arrangements with a quadratic minimal syzygy. We also obtain that the dual points lie on an interesting variety, though its description is not even close to the nice combinatorial case of the linear syzygy. Nevertheless, using this description we are able to classify up to a change of coordinates all rank 3 hyperplane arrangements having a quadratic minimal syzygy on their Jacobian ideal (Theorem 2.4). We end with a different and very interesting proof of Theorem 2.4 suggested by one of the anonymous referees, and with a question asking if it is possible to obtain a similar classification but for higher rank arrangements with a quadratic minimal logarithmic derivation.

2. Arrangements with low degree logarithmic derivations

2.1. Linear logarithmic derivations

Dropping the freeness condition which is not necessary in our study, [2, Proposition 4.29(3)] shows the following: If $\mathcal{A}$ is an arrangement with e_1 linearly independent degree 1 logarithmic derivations (including θ_E), then $\mathcal{A}$ is a direct product of e_1 irreducible arrangements.

One can obtain the same result, with a different interpretation of e_1, by studying the points dual to the hyperplanes of $\mathcal{A}$ in the following manner. Keeping the notations from the beginning of Introduction, let us assume that $\mathcal{A}$ has a linear logarithmic derivation, not a constant multiple of θ_E:

$$\theta = L_1\partial_1 + \cdots + L_k\partial_k,$$

where L_j are some linear forms in R. Because $\theta(x_i) = a_ix_i$, $i = 1, \dots, k$ for some constants $a_i \in \mathbb{K}$, then

$$L_i = a_ix_i, \quad i = 1, \dots, k,$$

and not all a_i's are equal to each-other (otherwise we would get a constant multiple of θ_E).

For $i \geq k+1$, suppose $\ell_i = p_{1,i}x_1 + \cdots + p_{k,i}x_k$, $p_{j,i} \in \mathbb{K}$. The logarithmic condition $\theta(\ell_i) = \lambda_i\ell_i$, $i \geq k+1$, $\lambda_i \in \mathbb{K}$ translates into

$$\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_k \end{bmatrix} \cdot \begin{bmatrix} p_{1,i} \\ \vdots \\ p_{k,i} \end{bmatrix} = \lambda_i \begin{bmatrix} p_{1,i} \\ \vdots \\ p_{k,i} \end{bmatrix}.$$

Therefore the points in $\mathbb{P}^{k-1}$ dual to the hyperplanes ℓ_i sit on the scheme with defining ideal I generated by the 2×2 minors of the matrix

$$\begin{bmatrix} a_1x_1 & a_2x_2 & \cdots & a_kx_k \\ x_1 & x_2 & \cdots & x_k \end{bmatrix}.$$

Obviously

$$I = \langle\{(a_i - a_j)x_ix_j : i \neq j\}\rangle,$$

and this is the edge (graph) ideal of a complete multipartite simple graph on vertices $1, \dots, k$; two vertices u and v belong to the same partition iff $a_u = a_v$.

For a simple graph G, a minimal vertex cover is a subset of vertices of G, minimal under inclusion, such that every edge of G has at least one vertex in this subset. By [4], since I is the edge ideal of a simple graph G (complete multipartite), all the minimal primes of I are generated by subsets of variables corresponding to the minimal vertex covers of G.[2] Also, since I is generated by square-free monomials, it must be a radical ideal, hence it is equal to the intersection of its minimal primes.

It is not difficult to show that if G is a complete multipartite graph with partition $P_1, \dots, P_s$, then the minimal vertex covers of G are $V(G) - P_i$, $i = 1, \dots, s$. So

$$I = I(G) = \cap_{i=1}^{s}\langle\{x_v : v \in V(G) - P_i\}\rangle.$$

[2] Greg Burnham, an REU student of Jessica Sidman, attributes this well known result to Rafael Villarreal, so we decided to use the same citation.