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Adjacent vertex distinguishing colorings by sum of sparse graphs

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ABSTRACT

A neighbor sum distinguishing edge-k-coloring, or nsd-k-coloring for short, of a graph G is a proper edge coloring of G with elements from $\{1, 2, \dots, k\}$ such that no pair of adjacent vertices meets the same sum of colors of G. The definition of this coloring makes sense for graphs containing no isolated edges (we call such graphs normal). Let $\mathrm{mad}(G)$ and $\Delta(G)$ be the maximum average degree and the maximum degree of a graph G, respectively. In this paper, we prove that every normal graph with $\Delta(G) \geq 5$ and $\mathrm{mad}(G) < 3$ admits an nsd-$(\Delta(G)+2)$-coloring. Our approach is based on the Combinatorial Nullstellensatz and the discharging method.

1. Introduction

The terminology and notation used but undefined in this paper can be found in [5]. Let G be a finite undirected simple graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, maximum degree and minimum degree of the graph G, respectively. Let $\mathrm{mad}(G) = \max\{\frac{2|E(H)|}{|V(H)|} \mid H \subset G\}$ be the maximum average degree of G. Set $[n] = \{1, 2, \dots, n\}$, where n is a non-negative integer. A graph G is *normal* if no connected component is isomorphic to K_2.

A proper edge coloring of a graph $G = (V(G), E(G))$ is an assignment of colors to the edges of G such that no two adjacent edges receive the same color. Let C be a finite set of colors and let $c : E(G) \to C$ be a proper edge coloring of G. The color set of a vertex $v \in V(G)$ with respect to c is the set $S(v)$ of colors of all edges incident to v, i.e., $S(v) = \{c(uv) \mid uv \in E(G)\}$. The coloring c is called a *neighbor set distinguishing edge-k-coloring*, or an *nd-k-coloring* of G for short if $S(u) \neq S(v)$ for each edge $uv \in E(G)$. The minimal number of colors necessary to construct such a coloring will be denoted by $\chi'_a(G)$, and called the *neighbor set distinguishing index* of G.

In 2002, Zhang et al. [22] proposed the following conjecture.

Conjecture 1.1 ([22]). *If G is a normal connected graph with at least 6 vertices, then $\chi'_a(G) \leq \Delta(G) + 2$.*

Hatami [11] showed that if G is a normal graph and $\Delta(G) > 10^{20}$, then $\chi'_a(G) \leq \Delta(G) + 300$. For more references, see [1,3,8,12]. Recently, in [13], Hocquard et al. proved the following result.

Theorem 1.2 ([13]). *Let G be a normal graph of maximum degree $\Delta(G) \geq 5$ and $\mathrm{mad}(G) < 3 - \frac{2}{\Delta(G)}$, then $\chi'_a(G) \leq \Delta(G) + 1$.*

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In [20], Wang et al. proved the following result.

Theorem 1.3 (*[20]*). *Let G be a normal graph of maximum degree* $\Delta(G) \geq 3$ *and* $\mathrm{mad}(G) < 3$*, then* $\chi'_a(G) \leq \Delta(G) + 2$.

The aim of this paper is to study a similar invariant. Consider $C = [k]$. In this case, we are allowed to consider the sum of colors at each vertex $v \in V(G)$, i.e., the number $w(v) = \sum_{e \ni v} c(e)$. The coloring c is called a *neighbor sum distinguishing edge-k-coloring*, or an *nsd-k-coloring* of G for short if $w(u) \neq w(v)$ for each edge $uv \in E(G)$. The least integer k necessary to construct such a coloring will be denoted by $\chi'_{\sum}(G)$, and called the *neighbor sum distinguishing index*.

Evidently, when searching for the neighbor sum (set) distinguishing index it is sufficient to restrict our attention to connected graphs. Observe also, that a graph G admits a neighbor sum (set) distinguishing edge-k-coloring if and only if G is normal. So, we shall consider only connected graphs with at least three vertices. Apparently, for any normal graph G, $\Delta(G) \leq \chi'(G) \leq \chi'_a(G) \leq \chi'_{\sum}(G)$, where $\chi'(G)$ is the chromatic index of G.

A motivation of our work on nsd-k-colorings comes from the study of the following famous conjecture.

Conjecture 1.4 (*[14], 1-2-3 Conjecture*). *If G is a graph with no component isomorphic to* K_2*, then the edges of G may be assigned weights from the set* $\{1, 2, 3\}$ *so that, for any adjacent vertices* $u, v \in V(G)$*, the sum of weights of the edges incident to u differs from the sum of weights of the edges incident to* v.

Recently, Flandrin et al. [9] studied the neighbor sum distinguishing colorings of cycles, trees, complete graphs and complete bipartite graphs. Based on these examples, they proposed the following conjecture.

Conjecture 1.5 (*[9]*). *If G is a connected graph on at least* 3 *vertices and* $G \neq C_5$*, then* $\chi'_{\sum}(G) \leq \Delta(G) + 2$.

Flandrin et al. [9] also proved that for each connected graph G with maximum degree $\Delta(G) \geq 2$, it holds that $\chi'_{\sum}(G) \leq \lceil \frac{7\Delta(G)-4}{2} \rceil$. Wang and Yan [21] improved this bound to $\lceil \frac{10\Delta(G)+2}{3} \rceil$. In [16], Przybyło proved that $ch'_{\sum}(G) \leq 2\Delta(G) + \mathrm{col}(G) - 1$, where $\mathrm{col}(G)$ is the coloring number of G. Lately, Przybyło and Wong [18] proved that $ch'_{\sum}(G) \leq \Delta(G) + 3\mathrm{col}(G) - 4$. The latest result thus far is that for any graph with $\Delta(G) \geq 2$, $\chi'_{\sum}(G) \leq (1 + o(1))\Delta(G)$ [17]. For planar graphs, Dong and Wang [6] proved that $\chi'_{\sum}(G) \leq \max\{2\Delta(G) + 1, 25\}$. Later this bound was improved to $\max\{\Delta(G) + 10, 25\}$ by Wang et al. in [19]. In [4], Bonamy and Przybyło proved that any normal planar graph with $\Delta(G) \geq 28$ satisfies $\chi'_{\sum}(G) \leq \Delta(G)+1$. Dong et al. also studied neighbor sum distinguishing colorings of sparse graphs in [7]. More precisely, they proved the following result there.

Theorem 1.6 (*[7]*). *Let G be a normal graph. If* $\mathrm{mad}(G) < \frac{5}{2}$ *and* $\Delta(G) \geq 5$*, then* $\chi'_{\sum}(G) \leq \Delta(G) + 1$.

In [10], Gao et al. improved the bound $\frac{5}{2}$ to $\frac{8}{3}$ and proved the following theorem.

Theorem 1.7. *Let G be a normal graph. If* $\mathrm{mad}(G) < \frac{8}{3}$ *and* $\Delta(G) \geq 5$*, then* $\chi'_{\sum}(G) \leq \Delta(G) + 1$.

In this paper, we will prove the following result via the Combinatorial Nullstellensatz and the discharging method.

Theorem 1.8. *Let G be a normal graph. If* $mad(G) < 3$ *and* $\Delta(G) \geq 5$*, then* $\chi'_{\sum}(G) \leq \Delta(G) + 2$.

Apparently, when $\Delta(G) \geq 5$, Theorem 1.8 implies Theorem 1.3.

2. Preliminaries

Let $P(x_1, x_2, \ldots, x_n)$ be a polynomial in n variables, where $n \geq 1$. We denote by $c_P(x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n})$ the coefficient of the monomial $x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$ in $P(x_1, x_2, \ldots, x_n)$, where k_i $(1 \leq i \leq n)$ is a non-negative integer.

In each configuration of all figures, the degree of each solid vertex is fixed and the degree of each hollow vertex is at least d, where d is the number of solid edges incident with this hollow vertex in the configuration; each solid edge must exist and the existence of every dotted edge cannot be guaranteed.

In the following we give several lemmas.

Lemma 2.1 (*[15]*). *Let* B_1, B_2 *be sets of integers with* $|B_1| = m \geq 2$ *and* $|B_2| = n \geq 2$*. Let* $B_3 = \{x + y \mid x \in B_1, y \in B_2, x \neq y\}$*. Then* $|B_3| \geq m + n - 3$*. Moreover, if* $B_1 \neq B_2$*, then* $|B_3| \geq m + n - 2$.

Lemma 2.2 (*[15]*). *Suppose* B_1 *is a set of integers and* $|B_1| = n$*. Let* $B_2 = \{\sum_{i=1}^{m} x_i \mid x_i \in B_1, x_i \neq x_j (i \neq j)\}$*, where* $m \leq n$*. Then* $|B_2| \geq mn - m^2 + 1$.