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# On acyclic edge-coloring of the complete bipartite graphs $K_{2p-1,2p-1}$ for odd prime $p$

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ABSTRACT

An acyclic edge-coloring of a graph is a proper edge-coloring without bichromatic (2-colored) cycles. The acyclic chromatic index of a graph $G$, denoted by $a'(G)$, is the least integer $k$ such that $G$ admits an acyclic edge-coloring using $k$ colors. Let $\Delta = \Delta(G)$ denote the maximum degree of a vertex in a graph $G$. A complete bipartite graph with $n$ vertices on each side is denoted by $K_{n,n}$. Basavaraju, Chandran and Kummini proved that $a'(K_{n,n}) \geq n + 2 = \Delta + 2$ when $n$ is odd. Basavaraju and Chandran showed that $a'(K_{p,p}) \leq p + 2$ which implies $a'(K_{p,p}) = p + 2 = \Delta + 2$ when $p$ is an odd prime, and the main tool in their proof is perfect 1-factorization of $K_{p,p}$. In this paper we study the case of $K_{2p-1,2p-1}$ which also possess perfect 1-factorization, where $p$ is odd prime. We show that $K_{2p-1,2p-1}$ admits an acyclic edge-coloring using $2p + 1$ colors and so we get $a'(K_{2p-1,2p-1}) = 2p + 1 = \Delta + 2$ when $p$ is an odd prime.



## 1. Introduction

Let $G = (V, E)$ be a finite and simple graph. A *proper edge-coloring* of $G$ is an assignment of colors to the edges so that no two adjacent edges have same color. So it is a map $\theta : E \to \mathcal{C}$ with $\theta(e) \neq \theta(f)$ for any adjacent edges $e, f \in E$, where $\mathcal{C}$ is the set of colors. The *chromatic index*, denoted by $\chi'(G)$, is the minimum number of colors needed to properly color the edges of $G$. A proper edge-coloring of $G$ is *acyclic* if there is no two colored cycle in $G$. The minimum number of colors required in an acyclic edge-coloring of $G$ is the *acyclic edge chromatic number* (also called *acyclic chromatic index*) and is denoted by $a'(G)$. The notion of acyclic coloring was first introduced by Grünbaum [7] in 1973, and the concept of acyclic edge-coloring was first studied by Fiamčík [6]. Let $\Delta = \Delta(G)$ be the maximum degree of a vertex in $G$. It is obvious that any proper edge-coloring requires at least $\Delta$ colors. Vizing [16] proved that there always exists a proper edge-coloring with $\Delta + 1$ colors. Since any acyclic edge coloring is proper, we must have $a'(G) \geq \chi'(G) \geq \Delta$. On the other hand, in 1978, Fiamčík [6] (also Alon, Sudakov and Zaks [1]) posed the following conjecture:

$$\text{for any graph } G,\ \ a'(G) \leq \Delta + 2. \tag{1}$$

In [1], it was proved that there exists a constant $c$ such that $a'(G) \leq \Delta + 2$ for any graph with girth is at least $c\Delta \log \Delta$. It was also proved in [1] that $a'(G) \leq \Delta + 2$ for almost all $\Delta$-regular graphs. Later Něsetřil and Wormald [15] improved this bound and showed that $a'(G) \leq \Delta + 1$ for a random regular graph $G$. In another direction, there have been many results

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giving upper bounds on $a'(G)$ for arbitrary graphs or a class of graphs. Recently, Ndreca et al. obtained $a'(G) \leq 9.62\Delta$ [14] which is currently the best upperbound for an arbitrary graph $G$. See [17, Section 3.3] for a nice account of recent results.

The above conjecture (1) was shown to be true for some special classes of graphs. Burnstein [5] showed that $a'(G) \leq 5$ when $\Delta = 3$. Hence the conjecture is true when $\Delta \leq 3$. Muthu, Narayanan and Subramanian proved that the conjecture holds true for grid-like graphs [11] and outerplanner graphs [12]. It has been observed that determining $a'(G)$ is a hard problem from both theoretical and algorithmic points of view [17, p. 2119]. In fact, we do not yet know the values of $a'(G)$ for some simple and highly structured graphs like complete graphs and complete bipartite graphs in general. Fortunately, we can get the exact value of $a'(G)$ for some cases of complete bipartite graphs, thanks to the perfect 1-factorization.

Let $K_{n,n}$ be the complete bipartite graph with $n$ vertices on each side. The complete bipartite graph $K_{n,n}$ is said to have a *perfect 1-factorization* if the edges of $K_{n,n}$ can be decomposed into $n$ disjoint perfect matchings such that the union of any two perfect matchings gives a Hamiltonian cycle. It is known that when $n+2 \in \{p, 2p-1, p^2\}$, where $p$ is an odd prime, or $n+2 < 50$ and odd, then $K_{n+2,n+2}$ has a perfect 1-factorization (see [4]). One can easily see that if $K_{n+2,n+2}$ has a perfect 1-factorization then $a'(K_{n,n}) \leq a'(K_{n+1,n+1}) \leq n+2$. And also we have

$$a'(K_{n,n}) \geq n+2 = \Delta + 2 \quad \text{when } n \text{ is odd}$$

due to Basavaraju, Chandran and Kummini [3]. Hence $a'(K_{n,n}) = n+2 = \Delta+2$ when $n+2 \in \{p, 2p-1, p^2\}$. The main idea here is to give different colors to the edges in different 1-factors in $K_{n+2,n+2}$, and removal of two vertices on each side and their associated edges gives the required edge-coloring of $K_{n,n}$. Similarly, by a result of Guldan [8, Corollary 1], we can also get $a'(K_{n+1,n+1}) = n+2 = \Delta+1$ when $n+2 \in \{p, 2p-1, p^2\}$. But a different approach is needed to deal with $K_{n+2,n+2}$ when $n+2 \in \{p, 2p-1, p^2\}$. In 2009, Basavaraju and Chandran [2] proved that $a'(K_{p,p}) = p+2 = \Delta+2$ for any odd prime $p$. The main tool in their approach is again perfect 1-factorization of $K_{p,p}$. In the remaining two cases, namely, $n+2 \in \{2p-1, p^2\}$ the value of $a'(K_{n+2,n+2})$ is not yet known. In this paper we study the case of $K_{2p-1,2p-1}$ which also possesses a perfect 1-factorization, where $p$ is odd prime. We show that $K_{2p-1,2p-1}$ admits an acyclic edge-coloring using $2p+1$ colors.

## 2. Our result

We state our main result as follows.

**Theorem 1.** *$a'(K_{2p-1,2p-1}) = 2p+1 = \Delta+2$, where $p$ is an odd prime.*

We follow the proof technique of [2] to present the proof of Theorem 1. Accordingly we first consider a perfect 1-factorization of $K_{2p-1,2p-1}$. Next we consider another perfect matching which satisfies certain conditions. Then we present an edge-coloring of $K_{2p-1,2p-1}$ using $2p+1$ colors and show that it is acyclic. In general, for odd $n$ if $K_{n,n}$ possesses a perfect 1-factorization, the difficulty is to identify a suitable perfect matching that can help to get an acyclic edge-coloring of $K_{n,n}$ using only $n+2$ colors. The main contribution of this paper is to identify such a suitable perfect matching and provide an acyclic edge-coloring of $K_{2p-1,2p-1}$ using $2p+1$ colors, where $p$ is an odd prime.

**Proof of Theorem 1.** We label the vertices of $K_{2p-1,2p-1}$ on each side with elements of the set $I = \{1, 2, \ldots, 2p-1\} = \mathbb{Z}_{2p} \backslash \{0\}$, and so a perfect matching (1-factor) can be represented by a permutation of the label set $I$. Let us now present a perfect 1-factorization of $K_{2p-1,2p-1}$ using permutations of the label set $I$. Let $M_j$ be the perfect matching corresponding to the permutation $\pi_j$ for $j \in I$ which we define below. In the definitions of $\pi_j$ below, $k \in I$ $(=\mathbb{Z}_{2p} \backslash \{0\})$ and the operations are understood to be done modulo $2p$ (that is in $\mathbb{Z}_{2p}$).

For $i = 1, 2, \ldots, p-1$, define

$$\pi_{2i}(k) = \begin{cases} 2i & \text{if } k = 2i \\ i+p & \text{if } k = i \\ i & \text{if } k = i+p \\ 2i-k & \text{otherwise.} \end{cases}$$

For $i = 0, 1, 2, \ldots, p-1$ and $i \neq \frac{p-1}{2}$, define

$$\pi_{2i+1}(k) = \begin{cases} 2i+1 & \text{if } k = 2i+1 \\ k-(2i+1) & \text{if } k \neq 2i+1 \text{ and } k \text{ is odd} \\ k+(2i+1) & \text{if } k \text{ is even.} \end{cases}$$

Also

$$\pi_p(k) = 2p - k = -k.$$

A perfect 1-factorization of $K_{2p-1,2p-1}$ is presented in [13, p. 31] applying Laufer's technique [10] on the formulation of perfect 1-factorization of the complete bipartite graph $K_{2p}$ given by Kobayashi [9]. The formulation presented above is a simple modification of the formulation given in [13, p. 31] to suit our representation. So the decomposition of the edges into