Discrete Mathematics 339 (2016) 78–83

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Intersecting faces of a simplicial complex via algebraic shifting

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ARTICLE INFO

Article history:
Received 12 September 2013
Received in revised form 23 July 2015
Accepted 24 July 2015
Available online 13 August 2015

Keywords:
Simplicial complex
Erdős–Ko–Rado theorem
Algebraic shifting
i-Near-cone
Sequentially Cohen–Macaulay simplicial complex

ABSTRACT

A family $\mathcal{A}$ of sets is *t-intersecting* if the size of the intersection of every pair of sets in $\mathcal{A}$ is at least t, and it is an *r-family* if every set in $\mathcal{A}$ has size r. A well-known theorem of Erdős, Ko, and Rado bounds the size of a t-intersecting r-family of subsets of an n-element set, or equivalently of $(r-1)$-dimensional faces of a simplex with n vertices. As a generalization of the Erdős–Ko–Rado theorem, Borg presented a conjecture concerning the size of a t-intersecting r-family of faces of an arbitrary simplicial complex. He proved his conjecture for shifted complexes. In this paper we give a new proof for this result based on work of Woodroofe. Using algebraic shifting we verify Borg's conjecture in the case of sequentially Cohen–Macaulay i-near-cones for $t = i$.

1. Introduction

Throughout this paper, the set of positive integers $\{1, 2, \ldots\}$ is denoted by $\mathbb{N}$. For $m, n \in \mathbb{N}$ with $m \leq n$, the set $\{i \in \mathbb{N} : m \leq i \leq n\}$ is denoted by $[m, n]$; for $m = 1$, we also write $[n]$.

Let t and r be natural numbers with $t \leq r$. A family $\mathcal{A}$ of sets is *t-intersecting* if the size of the intersection of every pair of sets in $\mathcal{A}$ is at least t, and it is an *r-family* if every set in $\mathcal{A}$ has size r.

A classical result in extremal set theory is the famous theorem of Erdős, Ko, and Rado [4]. There are several interesting proofs and generalizations of this theorem. For a nice survey on this topic we refer the reader to [2]. The Erdős–Ko–Rado Theorem asserts that the largest possible t-intersecting r-families of subsets of $[n]$ are the families of all r-subsets containing some fixed t-subset of points whenever n is sufficiently large with respect to t and r (whenever $n \geq n_0(t, r)$, where we use $n_0(t, r)$ to denote the least integer for which the theorem is valid). A more precise statement of this result was proved over a number of years by Frankl and Wilson [5,11], following a conjecture by Erdős. Indeed, they proved the following theorem.

Theorem 1.1. *Let t and r be natural numbers with $t \leq r$. If $n \geq (t+1)(r-t+1)$ and $\mathcal{A}$ is a t-intersecting r-family of subsets of $[n]$, then $|\mathcal{A}| \leq \binom{n-t}{r-t}$.*

In this paper we consider a generalization of Theorem 1.1 for simplicial complexes. Let us start with some preliminaries from simplicial complexes.

A simplicial complex Δ on the set of vertices $V(\Delta)$ is a collection of subsets of $[n]$ that is closed under taking subsets; that is, if $F \in \Delta$ and $F' \subseteq F$, then also $F' \in \Delta$. Every element $F \in \Delta$ is called a *face* of Δ. The *dimension* of a face F is defined to be $|F| - 1$. The *dimension* of Δ, which is denoted by $\dim \Delta$, is defined to be $d - 1$, where $d = \max\{|F| : F \in \Delta\}$. A *facet* of

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http://dx.doi.org/10.1016/j.disc.2015.07.014
0012-365X/

Δ is a maximal face of Δ with respect to inclusion. Let $\mathcal{F}(\Delta)$ denote the set of facets of Δ. It is clear that $\mathcal{F}(\Delta)$ determines Δ. When $\mathcal{F}(\Delta) = \{F_1, \dots, F_m\}$, we write $\Delta = \langle F_1, \dots, F_m\rangle$ and say that Δ is *generated by* $F_1, \dots, F_m$. A simplicial complex Δ is *pure* if all facets of Δ have the same size. The *link* of Δ with respect to a face $\sigma \in \Delta$, denoted by $\mathrm{lk}_\Delta(\sigma)$, is the simplicial complex

$$\mathrm{lk}_\Delta(\sigma) = \{\tau \subseteq [n] \setminus \sigma : \tau \cup \sigma \in \Delta\}$$

and the *anti-star* of Δ with respect to a face $\sigma \in \Delta$, denoted by $\mathrm{ast}_\Delta(\sigma)$, is the simplicial complex

$$\mathrm{ast}_\Delta(\sigma) = \{\tau \in \Delta : \tau \cap \sigma = \emptyset\}.$$

Also, for every integer $s \geq 0$ we define

$$\Delta_{(s)} = \{\sigma \in \Delta : |\sigma| = s\}.$$

Let Δ be a simplicial complex. The simplicial complex $\Delta^{(i)} := \{F \in \Delta : \dim F \leq i\}$ is the *i-skeleton* of Δ. Also, the simplicial complex $\Delta^{[i]} := \langle F \in \Delta : \dim F = i\rangle$ is the *i-pure skeleton* of Δ.

A face of Δ of size r is an *r-face* of Δ. Let f_r denote the number of r-faces of Δ. The sequence $(f_0, f_1, \dots, \dots, f_d)$ is called the *f-vector* of Δ.

Note. Many authors define an r-face to be a face with dimension r. We follow Swartz [10] and Woodroofe [12] in considering an r-face to be a face with size r (rather than dimension r).

One of the connections between combinatorics and commutative algebra is via rings constructed from the combinatorial objects. Let R be the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ in n variables over a field $\mathbb{K}$, and let Δ be a simplicial complex on $[n]$. For every subset $F \subseteq [n]$, we set $x_F = \prod_{i\in F} x_i$. The *Stanley–Reisner ideal of* Δ *over* $\mathbb{K}$ is the ideal I_Δ of R that is generated by those squarefree monomials x_F with $F \notin \Delta$. In other words, $I_\Delta = \langle x_F : F \in \mathcal{N}(\Delta)\rangle$, where $\mathcal{N}(\Delta)$ denotes the set of minimal nonfaces of Δ with respect to inclusion. The *Stanley–Reisner ring of* Δ *over* $\mathbb{K}$, denoted by $\mathbb{K}[\Delta]$, is defined by $\mathbb{K}[\Delta] = R/I_\Delta$.

We recall a notion from commutative algebra. Let $R = \mathbb{K}[x_1, \dots, x_n]$, and let M be a nonzero finitely generated R-module. We say that M is *Cohen–Macaulay* if for every prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, the equality depth $M_\mathfrak{p} = \dim M_\mathfrak{p}$ holds true. We say that a simplicial complex Δ is *Cohen–Macaulay over a field* $\mathbb{K}$ if the Stanley–Reisner ring $\mathbb{K}[\Delta]$ of Δ is Cohen–Macaulay. A well-known result of Reisner says that a simplicial complex Δ is Cohen–Macaulay over $\mathbb{K}$ if and only if for every $F \in \Delta$ and for i less than $\dim(\mathrm{lk}_\Delta(F))$, it holds that $\widetilde{H}_i(\mathrm{lk}_\Delta(F); \mathbb{K}) = 0$, where $\widetilde{H}_i(\Delta; \mathbb{K})$ denotes the simplicial homology of Δ with coefficients in $\mathbb{K}$. We say that a simplicial complex Δ is *sequentially Cohen–Macaulay over a field* $\mathbb{K}$ if every pure skeleton of Δ is Cohen–Macaulay over $\mathbb{K}$.

Woodroofe [12] defined the *depth* of Δ over $\mathbb{K}$ as

$$\mathrm{depth}_\mathbb{K}\Delta = \max\{\ell : \Delta^{(\ell)} \text{ is Cohen–Macaulay over } \mathbb{K}\}.$$

We note that $\mathrm{depth}_\mathbb{K}\Delta$ is at most the minimum facet dimension of Δ, and equality holds if Δ is sequentially Cohen–Macaulay over $\mathbb{K}$.

We restate Theorem 1.1 using the language of simplicial complexes:

Theorem 1.2. *Let t and r be natural numbers with $t \leq r$. If $n \geq (t+1)(r-t+1)$ and $\mathcal{A}$ is a t-intersecting r-family of faces of a simplex Δ with n vertices, then $|\mathcal{A}| \leq f_{r-t}(\mathrm{lk}_\Delta\sigma)$, where σ is a t-face of Δ.*

Definition 1.3. A simplicial complex Δ is *(t, r)-EKR* if every t-intersecting r-family $\mathcal{A}$ of faces of Δ satisfies $|\mathcal{A}| \leq \max f_{r-t}(\mathrm{lk}_\Delta\sigma)$, where the maximum is taken over all t-faces σ of Δ. Equivalently, Δ is (t, r)-EKR if the set of all r-faces containing some t-face σ has maximal size among all t-intersecting families of r-faces.

As a generalization of Theorem 1.2, Borg conjectured that:

Conjecture 1.4 (*[1, Conjecture 2.6]*). *Let t and r be natural numbers with $t \leq r$. Let Δ be a simplicial complex having minimal facet size $k \geq (t+1)(r-t+1)$, and suppose that $S \neq \emptyset$ is a subset of $[t, r]$. If $\mathcal{A}$ is a t-intersecting family of faces of Δ with $\mathcal{A} \subseteq \bigcup_{s\in S} \Delta_{(s)}$, then*

$$|\mathcal{A}| \leq \max \sum_{s\in S} f_{s-t}(\mathrm{lk}_\Delta\sigma), \qquad (*)$$

where the maximum is taken over all t-faces σ of Δ.

Borg proved Conjecture 1.4 for shifted complexes [1, Theorem 2.7]. Using algebraic shifting, Woodroofe gave a new proof for [1, Theorem 2.7] in the special case $t = 1$ and $S = \{r\}$ [12, Lemma 3.1]. In this paper we extend Woodroofe's proof and give a new proof for [1, Theorem 2.7] using algebraic shifting (Corollary 2.2). Woodroofe also proved that Conjecture 1.4 is true for sequentially Cohen–Macaulay near-cones in the special case $t = 1$ and $S = \{r\}$ [12, Corollary 3.4]. We also generalize this result and prove that Conjecture 1.4 is true for every sequentially Cohen–Macaulay i-near-cone in the case $t = i$ (Corollary 3.9).