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Some properties of vertex-oblique graphs

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ABSTRACT

The *type* $t_G(v)$ *of a vertex* $v \in V(G)$ is the ordered degree-sequence $(d_1, \dots, d_{d_G(v)})$ of the vertices adjacent with v, where $d_1 \leq \cdots \leq d_{d_G(v)}$. A graph G is called *vertex-oblique* if it contains no two vertices of the same type. In this paper we show that for reals a, b the class of vertex-oblique graphs G for which $|E(G)| \leq a|V(G)| + b$ holds is finite when $a \leq 1$ and infinite when $a \geq 2$. Apart from one missing interval, it solves the following problem posed by Schreyer et al. (2007): *How many graphs of bounded average degree are vertex-oblique?* Furthermore we obtain the tight upper bound on the independence and clique numbers of vertex-oblique graphs as a function of the number of vertices. In addition we prove that the lexicographic product of two graphs is vertex-oblique if and only if both of its factors are vertex-oblique.

1. Introduction

The regularity of a graph is uniquely understandable by all graph theory researchers. It is still an open question, however, how one should define the irregularity of a graph. This question seems to have been addressed first by Alavi et al. in [1]. It originates from the fact that each graph with more than one vertex has at least two vertices of the same degree. This means that no graph cannot be irregular globally. This observation has constituted a separate line of interest dealing with local irregularity of graphs. During the years many different measures of local irregularity of graphs and many definitions of graphs that are extremal with respect to these measures were introduced in the literature. In this context vertex-oblique graphs were investigated [2–4] as a consequence of the study on asymmetric polyhedral graphs. Independently of those works, the authors of this paper started to study vertex-oblique graphs by the inspiration of two concepts: highly irregular graphs defined in [1] and graphs in which the open neighbourhoods of every two vertices are non-isomorphic, analysed in [5].

Throughout this paper, we consider finite and undirected graphs G with vertex set $V(G)$ and edge set $E(G)$, without loops and multiple edges. For $v \in V(G)$ the symbols $N_G(v)$, $d_G(v)$, $\Delta(G)$, $\delta(G)$, $\alpha(G)$, and $\omega(G)$ stand for the open neighbourhood of v in G, the degree of v in G, the maximum and the minimum degree of G, the independence number of G, and the clique number of G, respectively. The *type* $t_G(v)$ *of vertex* $v \in V(G)$ is the ordered degree-sequence $(d_1, \dots, d_{d_G(v)})$ of the vertices adjacent with v, where $d_1 \leq \cdots \leq d_{d_G(v)}$. A graph G is called *vertex-oblique* if it contains no two vertices of the same type.

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In this paper we investigate the properties of vertex-oblique graphs from several points of view. First we show in Theorem 1 that for reals a, b the class

$$\mathcal{F}(a, b) = \{G : \ |E(G)| \leq a|V(G)| + b \text{ and } G \text{ is vertex-oblique}\}$$

is finite for $a \leq 1$ and infinite for $a \geq 2$. It solves, apart from the gap between 1 and 2, the problem posed in [4] concerning the average degree of vertex-oblique graphs. Then, in Theorem 2 and Corollary 3, we obtain the tight upper bound on the independence and clique numbers of a vertex-oblique graph in terms of its order. Finally we prove that the lexicographic product of two graphs is vertex-oblique if and only if both of its factors are vertex-oblique (Theorem 3).

In general, we follow the notation and terminology of [6]. In particular, $G - e$ and $G - v$ mean the graph obtained from G by removing the edge $e \in E(G)$ and by removing the vertex $v \in V(G)$ and its incident edges, respectively. A $(u - v)$-path $(u, v \in V(G))$ is a path in G, represented by the sequence of its vertices without repetitions, whose ends are u and v. The distance $d_G(u, v)$ between two vertices u, v in a graph G is the length of the shortest $(u-v)$-path in G (the number of vertices in this path minus 1).

The symbols $\mathbb{N}$ and $\mathbb{N}_0$ stand for the set of positive and nonnegative integers, respectively. For $k \in \mathbb{N}$ we adopt the convention $[k] = \{1, \ldots, k\}$.

2. Average degree of vertex-oblique graphs

As a starting point of this section we put the open problem given in [4].

Problem 1. Are there infinitely many vertex-oblique graphs with bounded average degree?

Because for each graph G the equality $2|E(G)| = \sum_{v \in V(G)} d_G(v)$ is satisfied, we may rephrase Problem 1 in a more detailed form.

Problem 2. Let a, b be given real numbers and let $\mathcal{F}(a, b)$ be the class of vertex-oblique graphs for which the condition $|E(G)| \leq a|V(G)| + b$ holds. Is the class $\mathcal{F}(a, b)$ infinite?

In the main result of this section, Theorem 1, we solve Problem 2 for all permissible b, giving a negative answer when $a \leq 1$, and an affirmative answer when $a \geq 2$. In particular, our investigation completely solves Problem 2 for a, b being integers. Moreover, as the theorem shows, in those two ranges of a the actual value of b is irrelevant. Before proving it, we state several lemmas.

The first simple lemma is the rooted analogue of the well-known fact that in every tree on more than two vertices there is a pendant path of length two or a vertex of degree at least three which has no more than one non-leaf neighbour.

Lemma 1. *Let T be a tree and $v \in V(T)$ be a vertex with $d_T(v) \geq 2$. For each connected component T^* of $T - v$ at least one of the following conditions holds:*

1. *$T^* = K_1$, or*
2. *there exists a vertex $x \in V(T^*)$ that is a leaf of T and has a neighbour y in T^* such that $d_T(y) = 2$, or*
3. *there exist two different vertices $v_1, v_2 \in V(T^*)$, such that v_1, v_2 are both leaves of T and $N_T(v_1) = N_T(v_2)$.*

Proof. Assume that $T^* \neq K_1$. Let $x \in V(T^*)$ be a vertex at largest distance from v in T. In other words, $d_T(v, x) = \max\{d_T(v, z) : \ z \in V(T^*)\}$. Obviously x is a leaf of T.

Let y be the unique neighbour of x. Since T^* is a tree, either y is adjacent to v in T or it has a unique neighbour in the direction of v. Thus, if $d_T(y) = 2$, then the condition 2 holds.

Suppose that $d_T(y) \geq 3$. Then y has at least one neighbour different from x, say z, such that $d_T(v, z) = d_T(v, x)$. The choice of x with respect to distances from v implies that also z is a leaf. Thus, the condition 3 is satisfied. ■

The *cyclomatic number* $\mu(G)$ of a graph G is defined as $|E(G)| - |V(G)| + c(G)$, where $c(G)$ is the number of connected components of G. A *bridge* of a graph G is an edge e satisfying $c(G) < c(G-e)$. By a *core* G_c of a graph G we mean its subgraph obtained by successive pruning away all vertices of degree one. If G is a tree, then any one of its vertices can serve as a core; whereas if G contains at least one cycle and is connected, then its core G_c is unique.

Let G be a graph and $v \in V(G_c)$. We denote by $G'(v)$ the subgraph induced in G by the set $(V(G) \setminus V(G_c)) \cup \{v\}$. The *tree of G with root v* is the connected component of $G'(v)$ containing v. A *tree of G* is an arbitrary tree of G with root v, where v is any vertex from $V(G_c)$. An example of a graph G, its core G_c, its graph $G'(v)$ and all the trees of G are given in Fig. 1.

Lemma 2. *Let G be a vertex-oblique graph and $v \in V(G_c)$. If T is a tree of G with root v and $T \neq K_1$, then all the following conditions hold.*

1. *$d_T(v) \leq 2$ and for each $x \in V(T) \setminus \{v\}$ we have $d_T(x) \leq 3$.*
2. *If $d_T(v) = 2$ ($d_T(x) = 3$ for $x \in V(T) \setminus \{v\}$), then one of the connected components of $T - v$ (one of the connected components of $T - x$, not containing v) is K_1 and the other one contains a vertex which is a leaf of T and its neighbour in T is of degree two.*
3. *T can contain at most one vertex of degree three.*