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The Robber Locating game

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ABSTRACT

We consider a game in which a cop searches for a moving robber on a graph using distance probes, studied by Carragher, Choi, Delcourt, Erickson and West, which is a slight variation on one introduced by Seager. Carragher et al. show that for any fixed graph G there is a winning strategy for the cop on the graph $G^{1/m}$ obtained by replacing each edge of G by a path of length m, if m is sufficiently large. They conjecture that the cop does not have a winning strategy on $K_n^{1/m}$ if $m < n$; we show that in fact the cop wins if and only if $m \geqslant n/2$, for all but a few small values of n. They also show that the robber can avoid capture on any graph of girth 3, 4 or 5, and ask whether there is any graph of girth 6 on which the cop wins. We show that there is, but that no such graph can be bipartite; in the process we give a counterexample for their conjecture that the set of graphs on which the cop wins is closed under the operation of subdividing edges. We also give a complete answer to the question of when the cop has a winning strategy on $K_{a,b}^{1/m}$.

1. Introduction

Pursuit and evasion games on graphs have been widely studied. Perhaps the most significant variant is the Cops and Robbers game, an instance of which is a graph G together with a fixed number of cops. The cops take up positions on vertices of G and a robber then starts on any unoccupied vertex. The cops and the robber take turns: the robber chooses either to remain at his current vertex or to move to any adjacent vertex, and then the cops simultaneously make moves of the same form. The game is played with perfect information, so that at any time each of the players knows the location of all others. The cops win if at any point one of them is at the same location as the robber. The cop number of a graph is the minimum number of cops required for the cops to have a winning strategy.

Early results on this game include those obtained by Nowakowski and Winkler [8], who categorised the graphs of cop number 1, and Aigner and Fromme [1], who showed that every planar graph has cop number at most 3. An important open problem is Meyniel's conjecture, published by Frankl [5], that the cop number of any n-vertex connected graph is at most $O(\sqrt{n})$—this has been shown to be true up to a $\log(n)$ factor for random graphs by Bollobás, Kun and Leader [2], following which Łuczak and Prałat improved the error term [12]. More recently several variations on the game have been analysed by Clarke and Nowakowski (e.g. [4]).

In this paper we consider the Robber Locating game, introduced in a slightly different form by Seager [9] and further studied by Carragher et al. [3]. In this game the robber initially occupies a vertex without disclosing which it is to the cop.

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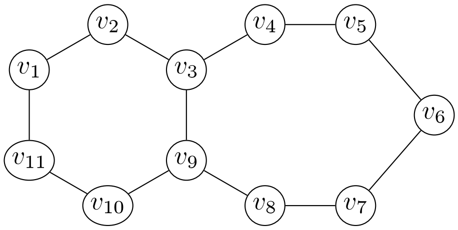

Fig. 1. Cycles of lengths 6 and 7 sharing an edge.

For ease of reading we shall refer to the cop as female and the robber as male. Each round consists of a move for the robber, in which he either moves to an adjacent vertex or stays where he is, followed by a probe of any vertex by the cop. When the cop probes a vertex she is told the current distance to the robber. In this setting the cop is not on the graph herself and can probe vertices without restriction; she wins if at any point she is able to determine the robber's current location.

Clearly the cop can win eventually with probability 1 on a finite graph against a robber who has no knowledge of her future moves, simply by probing random vertices until she hits the current location of the robber. This naturally leads to a different emphasis: we consider the question of whether the cop has a strategy that is guaranteed to win in bounded time, or equivalently whether she can catch an omniscient robber. We say that a graph is *locatable* if such a strategy exists and *non-locatable* otherwise. Noting that the cop will in general track the set of vertices that the robber could be at, with slight abuse of notation we allow ourselves to say he is *in a set* if he is known to be at one of the vertices comprising that set.

In the game as introduced by Seager there was an additional rule that the robber cannot move to the vertex probed in the previous round (the *no-backtrack condition*). Carragher et al. considered the game without this restriction, as do we. A similar game phrased in terms of a cat and a mouse, in which the cat wins only if it probes the current location of the mouse and receives no information otherwise, but the mouse must move at each turn, was recently analysed by one of the authors [7].

Given a graph G and a positive integer m, Carragher et al. [3] wrote $G^{1/m}$ for the graph obtained by replacing each edge of G by a path of length m through new vertices. Each such path is called a *thread*, and an *branch vertex* in $G^{1/m}$ is a vertex that corresponds to a vertex of G. The *span* of a branch vertex consists of all the vertices at distance less than m from it, which includes the vertices along the threads leaving that vertex but not the far endpoints of those threads. The main result of [3] is that $G^{1/m}$ is locatable when $m \geqslant \min\{|V(G)|, 1 + \max\{\mu(G) + 2^{\mu(G)}, \Delta(G)\}\}$, where $\mu(G)$ is the metric dimension of G. The *metric dimension* of a graph G, introduced independently by Slater [11] and by Harary and Melter [6], is the minimum size of a set S of vertices such that for every $x, y \in V(G)$ with $x \neq y$ there is some $z \in S$ with $d(x, z) \neq d(y, z)$.

Further, Carragher et al. [3] showed that for the complete bipartite graph $K_{a,b}$, the inequality $m \geqslant \max\{a, b\}$ is sufficient for $K_{a,b}^{1/m}$ to be locatable. They asked whether this is necessary. We show that in fact $m \geqslant (\min\{a, b\} - 1)$ is necessary and sufficient if $\min\{a, b\} \geqslant 4$, and $m \geqslant \min\{a, b\}$ is necessary and sufficient if $\min\{a, b\} \leqslant 3$, classifying all subdivided complete bipartite graphs. They also conjectured that their bound is tight for complete graphs, i.e. that $K_n^{1/m}$ is locatable if and only if $m \geqslant n$. We show that in fact, except for a few small values of n, the actual threshold is $n/2$.

They also proved that no graph of girth 3, 4 or 5 is locatable. The cycle C_6 is non-locatable, and so they asked whether there is a locatable graph of girth 6. We give an example of such a graph but show that no bipartite graph of girth 6 is locatable. In the process we give a counterexample to their conjecture that if G is locatable then so is any graph obtained by subdividing a single edge of G. We are grateful to our reviewers who drew our attention to a paper by Seager [10], announced while this paper was undergoing the review process, in which she independently also proved the result on graphs of girth 6.

2. Graphs of girth 6

In this section we first give an example of a locatable graph of girth 6 together with an explicit strategy for the cop. Define H to be the graph obtained from the cycle $v_1v_2 \cdots v_{11}$ by adding the edge v_3v_9. The graph consists of a 6-cycle and a 7-cycle with an edge in common. We include an illustration of H in Fig. 1.

Theorem 1. *The graph H as defined above is locatable.*

Proof. We first give several situations from which the cop can either win or reduce to an earlier situation; we then show how she can reach a winning situation. Here, when we say '*the robber is know to be at*', we refer to the set of vertices that are possible locations just after the robber's reply to the cop's probe, following which the robber makes a move before the next probe.

(i) If the robber is known to be at v_2 or v_4, then the cop wins by probing v_1.
(ii) If the robber is known to be at v_3 or v_4, then the cop probes v_9, winning or reducing to (i).
(iii) If the robber is known to be at v_3 or v_8, then the cop probes v_7, winning or reducing to (ii).
(iv) If the robber is known to be at v_3 or v_9, then the cop probes v_{10}, winning or reducing to (i) or (iii).
(v) If the robber is known to be at v_4 or v_5, then the cop wins by probing v_6.