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A note on packing of graphic n-tuples ☆

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ABSTRACT

A nonnegative integer n-tuple $(d_1, \dots, d_n)$ (not necessarily monotone) is *graphic* if there is a simple graph G with the vertex set $\{v_1, \dots, v_n\}$ in which the degree of v_i is d_i. Graphic n-tuples $(d_1^{(1)}, \dots, d_n^{(1)})$ and $(d_1^{(2)}, \dots, d_n^{(2)})$ *pack* if there are edge-disjoint n-vertex graphs G_1 and G_2 with the same vertex set $\{v_1, \dots, v_n\}$ such that $d_{G_1}(v_i) = d_i^{(1)}$ and $d_{G_2}(v_i) = d_i^{(2)}$ for all i. Let $\Delta(\pi)$ and $\delta(\pi)$ denote the largest and smallest entries in n-tuple π respectively. For graphic n-tuples π_1 and π_2, Busch et al. (2012) proved that if $\Delta(\pi_1+\pi_2) \le \sqrt{2\delta(\pi_1+\pi_2)n} - (\delta(\pi_1+\pi_2)-1)$ (strict inequality when $\delta(\pi_1+\pi_2) = 1$), then π_1 and π_2 pack. As a more direct analogue to the Sauer–Spencer Theorem, Busch et al. conjectured that if $\delta(\pi_1+\pi_2) \ge 1$ and the product of corresponding entries in π_1 and π_2 is always less than $\frac{n}{2}$, then π_1 and π_2 pack. In this paper, we prove that if $\delta(\pi_1) \ge 1$, π_2 is an almost k-regular graphic n-tuple with $k \ge 1$, and the product of corresponding entries in π_1 and π_2 is always less than $\frac{n}{2}$, then π_1 and π_2 pack. This is an interesting variation of Kundu's Theorem. Combining this result with a counterexample to the above conjecture of Busch et al., we present a slight modification of this conjecture as follows: if $\delta(\pi_1) \ge 1$, $\delta(\pi_2) \ge 1$ and the product of corresponding entries in π_1 and π_2 is always less than $\frac{n}{2}$, then π_1 and π_2 pack.

1. Introduction

A nonnegative integer n-tuple $(d_1, \dots, d_n)$ (not necessarily monotone) is *graphic* if there is a simple graph G with vertex set $\{v_1, \dots, v_n\}$ such that $d_G(v_i) = d_i$. Such a graph G is a *realization* of $(d_1, \dots, d_n)$. The set of all graphic n-tuples is denoted by $\mathbf{F_n}$. The largest and smallest entries in an n-tuple π are denoted by $\Delta(\pi)$ and $\delta(\pi)$, respectively. Two n-vertex graphs G_1 and G_2 *pack* if they can be expressed as edge-disjoint subgraphs of the complete graph K_n. Let $\pi_1 = (d_1^{(1)}, \dots, d_n^{(1)}) \in \mathbf{F_n}$ and $\pi_2 = (d_1^{(2)}, \dots, d_n^{(2)}) \in \mathbf{F_n}$. We say that π_1 and π_2 *pack* if there exist edge-disjoint graphs G_1 and G_2 with the same vertex set $\{v_1, \dots, v_n\}$ such that $d_{G_1}(v_i) = d_i^{(1)}$ and $d_{G_2}(v_i) = d_i^{(2)}$ for all i.

Interestingly, the problem of packing graphic n-tuples has concrete applications to discrete imaging science. Of particular interest here is *discrete tomography*, which uses low-dimensional projections to reconstruct discrete objects, such as the atomic structure of crystalline lattices and other polyatomic structures.

Numerous papers (cf. [3–5,7,8]) study the *k-color Tomography Problem*, in which the goal is to color the entries of an $m \times n$ matrix using k colors so that each row and column receives a prescribed number of entries of each color. The colors represent different types of atoms appearing in a crystal, and the number of times an atom appears in a given row or column is generally obtained using high resolution transmission electron microscopes [9,13]. This is precisely the problem of packing the degree sequences of k bipartite graphs with partite sets of size m and n.

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The condition that $\pi_1+\pi_2$ is graphic is obviously necessary for π_1 and π_2 to pack, but Busch et al. [1] showed that it is not sufficient. Therefore, it would be interesting to find some sufficient conditions for π_1 and π_2 to pack. In 1978, Sauer and Spencer [12] published the classical theorem that n-vertex graphs G_1 and G_2 pack if $\Delta(G_1)\Delta(G_2)<\frac{n}{2}$, where $\Delta(G)$ denotes the maximum vertex degree in G. Busch et al. [1] investigated an analogue of the Sauer–Spencer Theorem for graphic n-tuples and obtained a sufficient condition for π_1 and π_2 to pack.

Theorem 1 (*Busch et al. [1]*)**.** *Let $\pi_1,\pi_2\in\mathbf{F_n}$. If*

$$\Delta(\pi_1+\pi_2)\le\sqrt{2\delta(\pi_1+\pi_2)n}-(\delta(\pi_1+\pi_2)-1),$$

then π_1 and π_2 pack, except that strict inequality is required for the inequality to be sufficient when $\delta(\pi_1+\pi_2)=1$.

Busch et al. [1] showed that the bound in Theorem 1 is sharp, that is, they constructed graphic n-tuples that do not pack when the maximum entry in the sum is larger by 1.

A nonnegative integer n-tuple $\pi=(d_1,\ldots,d_n)$ is *an almost k-regular n-tuple* if $\delta(\pi)=k\ge 0$, $\Delta(\pi)\le n-1$, and $\Delta(\pi)-\delta(\pi)\le 1$. By Lemma 1 of [2], if $\pi=(d_1,\ldots,d_n)$ is an almost k-regular n-tuple and $\sum_{i=1}^{n}d_i$ is even, then π is graphic. Kundu's Theorem [10], published in 1973 and proved independently by Lovász [11] at about the same time, characterizes when $\pi\in\mathbf{F_n}$ has a realization containing a spanning subgraph that is almost k-regular. In the language of packing, the result is equivalent to the statement that if $\pi_1\in\mathbf{F_n}$ and each term in π_2 is k or $k+1$ with $k\ge 0$, then π_1 and π_2 pack if $\pi_1+\pi_2\in\mathbf{F_n}$. As an interesting variation of Kundu's Theorem, we obtain a new sufficient condition for π_1 and π_2 to pack in this paper, as follows.

Theorem 2. *Let $\pi_1,\pi_2\in\mathbf{F_n}$. If $\delta(\pi_1)\ge 1$, π_2 is an almost k-regular graphic n-tuple with $k\ge 1$, and the product of corresponding entries in π_1 and π_2 is always less than $\frac{n}{2}$, then π_1 and π_2 pack.*

Busch et al. [1] also conjectured the stronger statement that π_1 and π_2 pack if $\delta(\pi_1+\pi_2)\ge 1$ and the product of corresponding terms is always less than $\frac{n}{2}$; this would be a more direct analogue of the Sauer–Spencer Theorem.

Conjecture 3 (*Busch et al. [1]*)**.** *Let $\pi_1,\pi_2\in\mathbf{F_n}$. If $\delta(\pi_1+\pi_2)\ge 1$ and the product of corresponding entries in π_1 and π_2 is always less than $\frac{n}{2}$, then π_1 and π_2 pack.*

A counterexample to Conjecture 3 can be found as follows. There will be two constructions, one for even n and one (slightly modified) for odd n. For positive integer ℓ with $\ell\ge 2$, choose positive integer m with $m\ge 2\ell^2-\ell$ and let $n=2m$. Let $\pi_1=((m-1)^{m-\ell},\ell^{m+\ell-2\ell^2},(\ell-1)^{2\ell^2})$ and $\pi_2=(1^{2(m-\ell-1)},0^{2\ell+2})$, where the symbol x^y stands for y consecutive terms x. From the Erdős–Gallai characterization of graphic n-tuples, it is easy to see that both π_1 and π_2 are graphic. However, $\pi_1+\pi_2=(m^{m-\ell},(\ell+1)^{m+\ell-2\ell^2},\ell^{2\ell^2-2\ell-2},(\ell-1)^{2\ell+2})$ is not graphic, by the strict inequality

$$\begin{aligned}m(m-\ell)&>(m-\ell)(m-\ell-1)+(\ell+1)(m+\ell-2\ell^2)+\ell(2\ell^2-2\ell-2)+(\ell-1)(2\ell+2)\\&=m(m-\ell)-2.\end{aligned}$$

Therefore, π_1 and π_2 do not pack. However, π_1 and π_2 satisfy $\delta(\pi_1+\pi_2)=\ell-1\ge 1$, and the product of corresponding entries in π_1 and π_2 is always less than $\frac{n}{2}$.

For positive integer ℓ with $\ell\ge 2$, choose positive integer m with $m\ge 2\ell^2-2\ell+1$ and let $n=2m-1$. Let $\pi_1=((m-1)^{m-\ell},\ell^{m+2\ell-1-2\ell^2},(\ell-1)^{2\ell^2-\ell})$ and $\pi_2=(1^{2(m-\ell-1)},0^{2\ell+1})$. From the Erdős–Gallai characterization of graphic n-tuples, it is easy to see that both π_1 and π_2 are graphic. However, $\pi_1+\pi_2=(m^{m-\ell},(\ell+1)^{m+2\ell-1-2\ell^2},\ell^{2\ell^2-3\ell-1},(\ell-1)^{2\ell+1})$ is not graphic, by the strict inequality

$$\begin{aligned}m(m-\ell)&>(m-\ell)(m-\ell-1)+(\ell+1)(m+2\ell-1-2\ell^2)+\ell(2\ell^2-3\ell-1)+(\ell-1)(2\ell+1)\\&=m(m-\ell)-2.\end{aligned}$$

Therefore, π_1 and π_2 do not pack. However, π_1 and π_2 satisfy $\delta(\pi_1+\pi_2)=\ell-1\ge 1$, and the product of corresponding entries in π_1 and π_2 is always less than $\frac{n}{2}$.

The above counterexample to Conjecture 3 shows the sharpness of the requirement that $\delta(\pi_1)\ge 1$ and $\delta(\pi_2)\ge 1$. Therefore, we modify Conjecture 3 slightly as follows.

Conjecture 4. *Let $\pi_1,\pi_2\in\mathbf{F_n}$. If $\delta(\pi_1)\ge 1$, $\delta(\pi_2)\ge 1$ and the product of corresponding entries in π_1 and π_2 is always less than $\frac{n}{2}$, then π_1 and π_2 pack.*

Theorem 2 proves a special case of Conjecture 4 when π_2 is an almost k-regular graphic n-tuple with $k\ge 1$.

2. Proof of Theorem 2

We will directly apply Kundu's Theorem [10] to prove Theorem 2. Chen [2] gave a short proof of Kundu's Theorem. In the language of packing, Kundu's Theorem is equivalent to the following.