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Integral trees with given nullity

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ABSTRACT

A graph is called integral if all eigenvalues of its adjacency matrix consist entirely of integers. We prove that for a given nullity more than 1, there are only finitely many integral trees. Integral trees with nullity at most 1 were already characterized by Watanabe and Brouwer. It is shown that integral trees with nullity 2 and 3 are unique.

1. Introduction

Throughout this article, all graphs are assumed to be finite and without loops or multiple edges. For a graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. The *order* of G is defined as $|V(G)|$. The *adjacency matrix* of G, denoted by $A(G)$, is a matrix whose entries indexed by $V(G) \times V(G)$ and the (u, v)-entry is 1 if u and v are adjacent and 0 otherwise. The *characteristic polynomial* of G, denoted by $\varphi(G; x)$, is the characteristic polynomial of $A(G)$. We will drop the indeterminate x for the simplicity of notation. The zeros of $\varphi(G)$ are called the *eigenvalues* of G. Note that $A(G)$ is a real symmetric matrix so that all eigenvalues of G are real numbers. We denote the eigenvalues of G in non-increasing order as $\lambda_1(G) \geqslant \cdots \geqslant \lambda_n(G)$, where n is the order of G. The graph G is said to be *integral* if all eigenvalues of G are integers. The *nullity* of G is defined as the nullity of $A(G)$, which is equal to the multiplicity of 0 as an eigenvalue of G. A large number of articles on nullity of graphs have been published. We refer the reader to see [9] and references therein for a survey on this topic.

The notion of integral graphs was first introduced in [10]. A lot of articles deal with integral graphs. We refer the reader to [1] for a comprehensive but rather old survey on the subject. Here, we are concerned with integral trees. These objects are extremely rare and hence very difficult to find. For a long time, it was an open question whether there exist integral trees with arbitrarily large diameter [13]. Recently, this question was affirmatively answered in [4,8], where the authors constructed integral trees for any diameter. It is well known that the tree on two vertices is the only integral tree with nullity zero [12]. Thereafter, Brouwer proved that any integral tree with nullity 1 is a subdivision of a star graph where the order of the star graph is a perfect square [2]. The latter result has motivated us to investigate integral trees from the 'nullity' point of view.

In this article, we prove that with a fixed nullity more than 1, there are only finitely many integral trees. We also characterize integral trees with nullity 2 and 3 showing that there is a unique integral tree with nullity 2 as well as a unique integral tree with nullity 3.

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2. Reduced trees

In this section we introduce 'reduced trees' and derive some properties of their spectrum. We shall use these properties in the next section to prove our finiteness result.

As usual, the *degree* of a vertex v of a graph G is the number of edges of G incident on v. A vertex of degree 1 is called *pendant* and a vertex adjacent to a pendant vertex is said to be *quasi-pendant*. Write P_n for the path graph of order n. For a vertex v of a graph G, we say that there are *k pendant P_2 at v* if removing v from G increases the number of P_2 components by k. A graph G is called *reduced* if there exists at most one pendant P_2 at each vertex of G.

We denote the multiplicity of λ as an eigenvalue of a graph G by $\mathrm{mult}(G; \lambda)$. We also denote the number of eigenvalues of G in the interval $(-1, 1)$ by $m(G)$. It is worth to mention that the eigenvalue spectrum of any bipartite graph is symmetric with respect to the origin [3, p. 6].

The following folklore fact, which is stated in [7, p. 49] as an exercise, shows that a reduced graph obtained from a graph G by removing some pendant P_2 has the same nullity as G.

Lemma 1. *Let G be a graph and $v \in V(G)$ be a pendant vertex. If u is the neighbor of v, then the nullities of G and $G - \{u, v\}$ are the same.*

The following result is immediately deduced from Lemma 1 and is proved in [6, Theorem 2].

Corollary 2. *The size of the maximum matching in a tree of order n with nullity h is $\frac{n-h}{2}$.*

The first and second statements of the following theorem are respectively obtained from the Cauchy interlacing theorem for symmetric matrices [3, Corollary 2.5.2] and the Perron–Frobenius theory of nonnegative matrices [3, Theorem 2.2.1].

Theorem 3. *If G is a graph of order n and H is an induced subgraph of G of order m, then $\lambda_{n-m+i}(G) \leqslant \lambda_i(H) \leqslant \lambda_i(G)$ for $i = 1, \ldots, m$. Moreover, if G is a connected graph and $G \neq H$, then $\lambda_1(H) < \lambda_1(G)$.*

As a consequence of Theorem 3, one readily deduces that $\lambda_1(G) > \lambda_2(G)$ for any connected graph G of order at least 2.

Lemma 4. *Let G be a graph and $v \in V(G)$ be a pendant vertex. If u is the neighbor of v, then $m(G - \{u, v\}) \leqslant m(G)$.*

Proof. Note that $m(G - \{u, v\}) = m(G - u) - 1$. Applying Theorem 3 for G and $G - u$, we see that $m(G - u) - 1 \leqslant m(G)$, implying the result. $\square$

The following lemma generalizes a result in [12].

Lemma 5. *The tree P_2 is the only tree with no eigenvalue in $(-1, 1)$.*

Proof. We have $m(P_1) = 1$. By induction on n, we will show for any tree T of order $n \geqslant 3$ that $m(T) \geqslant 1$. Let v be a pendant vertex in a tree T and v' be its neighbor. If $T_v = T - \{v, v'\}$ has a connected component other than P_2, then it follows from Lemma 4, $m(P_1) = 1$, and the induction hypothesis that $m(T) \geqslant m(T_v) \geqslant 1$, as desired. Otherwise, all the connected components of T_v must be P_2. Indeed, we may assume that this property holds for each pendant vertex v of T. This forces that $T = P_4$. But $m(P_4) = 2$ by [5, Table 2], completing the proof. $\square$

Theorem 6. *For any nonnegative integer k, there are finitely many reduced trees with exactly k eigenvalues in $(-1, 1)$.*

Proof. We prove the assertion by induction on k. By Lemma 5, we may assume that $k \geqslant 1$. Let T be a reduced tree with $m(T) = k$. First suppose that there exists $v \in V(T)$ such that three of the connected components $T_1, \ldots, T_d$ of $T - v$ are not P_2. From Theorem 3, $m(T - v) \leqslant m(T) + 1$. Since T is reduced, at most one of $T_1, \ldots, T_d$ is P_2. Hence, Lemma 5 yields that $d - 1 \leqslant \sum_{i=1}^{d} m(T_i) \leqslant k + 1$ and $m(T_i) + 2 \leqslant \sum_{i=1}^{d} m(T_i) \leqslant k + 1$ for $i = 1, \ldots, d$. It follows that $d \leqslant k + 2$ and $m(T_i) \leqslant k - 1$ for $i = 1, \ldots, d$. Note that if some T_i is not reduced, then it has exactly one vertex with more than one pendant P_2 and such a vertex has exactly two pendant P_2. By the induction hypothesis, the number of reduced trees F with $m(F) \leqslant k - 1$ is finite and thus there are only finitely many ways of choosing $T_1, \ldots, T_d$. Since $d \leqslant k + 2$, the result follows. Now suppose otherwise. This means that any vertex of T is of degree at most 3 and all vertices of degree 3 of T have a pendant P_2. Hence, T is obtained from a path graph P_t by attaching one pendant P_2 at some vertices of degree 2 in P_t. Moreover, it follows from Lemma 4 that $m(P_t) \leqslant m(T)$. We know from [3, p. 9] that $\lambda_i(P_t) = 2\cos\frac{\pi\ell}{t+1}$ for $\ell = 1, \ldots, t$. Therefore, $m(P_t) \geqslant \frac{t-2}{3}$ and so $t \leqslant 3k + 2$. This completes the proof. $\square$

For later use, we need the following refinement of Lemma 4.

Lemma 7. *Let T be a tree with at least one pendant P_2 at $v \in V(T)$. Then increasing the number of pendant P_2 at v by one, leaves the number of eigenvalues in $(-1, 1)$ unchanged and increases the multiplicity of 1 by one.*