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Odd complete minors in even embeddings on surfaces

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ABSTRACT

In this paper, we study the *odd* K_m-minor problem in even embeddings on surfaces. We first establish a general theory for even embeddings with odd K_m-minors. Given an integer m we show that for every surface F^2 of sufficiently high genus there exists a constant $N = N(F^2)$ so that every non-bipartite even embedding on F^2 with representativity at least N contains an odd K_m as a minor. In the second part we prove that every 19-representative non-bipartite even embedding in an arbitrary orientable surface of genus ≥ 1 has an odd K_5-minor.

1. Introduction

A *contraction* of an edge e in a graph G is the operation which removes the edge e itself and identifies the two endvertices of e. We say that a graph H is a *minor* of G if there exists a sequence of graphs $G = G_1, G_2, \ldots, G_\ell = H$ such that for $i \in \{1, \ldots, \ell - 1\}$ the graph G_{i+1} is obtained from G_i by either an edge contraction, an edge deletion or a removal of an isolated vertex. (If both G and H are connected, then the third operation is not needed.) In this case, we say that G has H as a minor or an H-*minor*. A k-*cycle* C is a cycle of length $|C| = k$; it is an *odd* cycle if k is odd and an *even* cycle if k is even. We denote the complete graph on m vertices with K_m.

The existence problem of a K_m-minor in a graph is an important problem, because it is (among others) related to a well-known *Hadwiger Conjecture* [5], which states that every graph with a K_{m+1}-minor is m-colorable. For $m \leq 4$, the conjecture trivially holds, and the case with $m = 4$ is equivalent to the Four Color Theorem through the Kuratowski–Wagner theorem [26] for a characterization of K_5-minor-free structures. The case with $m = 5$ has been solved in [24], but the case when $m \geq 6$ is still open.

A *surface* is a compact 2-dimensional manifold without boundary. By the classification of surfaces, every surface is homeomorphic to an *orientable surface* of genus $g \geq 0$, denoted by $\mathbb{S}_g$, or a *nonorientable surface* of genus $k \geq 1$, denoted by $\mathbb{N}_k$. For a surface F^2, let $g(F^2)$ denote the *Euler genus* of F^2, that is, $g(F^2) = 2 - \chi(F^2)$, where $\chi(F^2)$ is the Euler characteristic of F^2. Let G be a *map* on a non-spherical surface F^2, that is, a fixed embedding of a graph with each face homeomorphic to an open 2-cell. For two maps G and H on the same surface F^2, we say that H is a *surface minor* of G if H is obtained from G by repeatedly contracting and/or deleting edges. (This definition is similar to the minor relation of two graphs, but in the surface minor relation, we note that in the process to obtain H from G, each of the graph operations is applied to maps on the surface without re-embedding graphs on the surface.) A simple closed curve λ is *contractible* on F^2 if λ can be transformed into a

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single point by a continuous transformation, and otherwise, λ is *essential* (also *noncontractible*). These definitions extend to cycles and also closed walks of embedded graphs. The *representativity* of G (also called *face-width* of G), denoted by $r(G)$, is the minimum number of intersecting points of G and λ on F^2, where λ ranges over all essential simple closed curves on F^2. We say G is *k-representative* if G has representativity at least k. It is easy to see that if a map H is a surface minor of G, then $r(H) \leq r(G)$.

The problem of finding K_m-minors has also been considered on surfaces. (However, we have to note that for any surface F^2, the number m for which K_m embeds in F^2 is bounded, where m is known as a *Heawood number*.) Fijavž and Mohar [3] proved that every 5-connected 3-representative map on the projective plane has a K_6-minor, and Mukae et al. and others characterized triangulations on low-genus surfaces with K_6-minors [11,14,15,18]. We explicitly state a theorem of Krakovski and Mohar.

Theorem 1 (*Krakovski and Mohar [12]*). *There exists an absolute constant N such that every N-representative graph on a non-spherical surface has a K_6-minor.*

They proved that for every non-spherical surface $N = 6$ suffices, and that 4 is the sufficient bound in the projective-planar case. Moreover, they claim it suffices to consider the number of intersecting points of graphs and only *nonseparating* simple closed curve on the surfaces, though the representativity should be defined for *all* essential simple closed curves. Note that Theorem 1 contrasts to the well known and powerful theorem of Robertson and Seymour:

Theorem 2 (*Robertson and Seymour [22]*). *Let H be a map on a non-spherical surface F^2. Then there exists an integer $N = N(F^2, H)$ such that every N-representative map on F^2 has an H-minor, up to homeomorphism.*

Clearly, if F^2 admits an embedding of G and does not admit an embedded H, then also G does not contain an H-minor. If, on the other hand, G embeds in F^2 and so does H, then Theorem 2 implies that, provided the representativity of (an embedding of) G in F^2 is large enough (with respect to F^2), then H is a minor of G.

Theorem 1, on the other hand, yields an absolute constant. One has to note that K_6 embeds in every non-spherical surface.

We shall consider the *odd minor* problem, a natural refinement of the usual minor problem, see, for example [4,10].

Let H be a graph with $V(H) = \{v_1, \ldots, v_m\}$. We can say that a graph G has an *H-minor* if G has m pairwise disjoint sub-trees $T_1, \ldots, T_m$ such that if $v_iv_j \in E(H)$, then G has an edge e_{ij} joining T_i and T_j. (This is equivalent to the minor relation given in the first paragraph.) We say that H is an *odd* minor of G if there is a 2-color-assignment $c : V(G) \to \{1, 2\}$ such that $c|_{V(T_i)} : V(T_i) \to \{1, 2\}$ is a proper coloring for each i, and that for each $e_{ij} = xy$, $c(x) = c(y)$.

It is easy to see that

(P1) if G has an odd H-minor and H has a cycle C, then we can find a unique cycle D in the subgraph

$$\left(\bigcup_{i=1}^{m} T_i\right) \cup \{e_{ij} : v_iv_j \in E(H)\}$$

of G such that D contracts to C and that $|D| \equiv |C| \pmod 2$,

since D is a cycle of G containing exactly $|C|$ monochromatic edges in the 2-color-assignment of G. We also call D the *model* of C.

It is also not difficult to argue that the odd-minor relation is transitive, as models of cycles do not change their parities. If G_1 is an odd minor of G_2, and the latter is an odd minor of G_3, then G_1 is also an odd minor of G_3.

The existence of an odd K_m-minor in G gives the information about the parity of cycle lengths in G. By the parity argument no bipartite graph has an odd K_m-minor for any $m \geq 3$, since K_3 itself contains an odd cycle. In this paper, we always suppose that $H = K_m$ with $m \geq 3$, and consider the existence problem of an odd K_m-minor.

Let G be an *even embedding* on a surface F^2, that is, a map of a simple graph on F^2 with each face bounded by an even cycle. Then G is a 2-connected map on F^2, and if F^2 is non-spherical, then G is also 2-representative. In particular, G is a *quadrangulation* if each face is bounded by a 4-cycle. The following are well-known fundamental properties of cycle lengths of an even embedding G on F^2:

(E1) If F^2 is the sphere, then G must be bipartite. On the other hand, every non-spherical surface admits non-bipartite even embeddings.
(E2) If C is a contractible closed walk, then C has even length.
(E3) If C and C' are closed walks of G freely homotopic on F^2, then $|C| \equiv |C'| \pmod 2$.

These special properties of even embeddings makes our problem interesting, and enable us to establish a theory for the existence of odd K_m-minors in graphs on surfaces.

Definition 3. For the sphere $\mathbb{S}_0$, the Klein bottle $\mathbb{N}_2$, and the double torus $\mathbb{S}_2$, let

$$m(\mathbb{S}_0) = 2, \qquad m(\mathbb{N}_2) = 4, \qquad m(\mathbb{S}_2) = 5.$$