Discrete Mathematics 339 (2016) 217–221

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

A note on the no-three-in-line problem on a torus

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ARTICLE INFO

Article history:
Received 25 September 2014
Received in revised form 5 August 2015
Accepted 8 August 2015
Available online 4 September 2015

Keywords:
Discrete torus
No-three-in-line problem
Chinese Remainder Theorem

ABSTRACT

In this paper we show that at most 2 gcd(m, n) points can be placed with no three in a line on an $m \times n$ discrete torus. In the situation when gcd(m, n) is a prime, we completely solve the problem.

1. Introduction

The no-three-in-line-problem [2] asks for the maximum number of points that can be placed in the $n \times n$ grid with no three points collinear. This question has been widely studied, but is still not resolved.

The obvious upper bound is $2n$ since one can put at most two points in each row. This bound is attained for many small cases, for details see [4] and [5]. In [7] the authors give a probabilistic argument to support the conjecture that for a large n this limit is unattainable.

As a lower bound, Erdös' construction (see [3]) shows that for p prime one can select p points with no three collinear. In [8] it is shown, that for p prime one can select $3(p-1)$ points from a $2p \times 2p$ grid with no three collinear.

In the literature we can find some extensions of the no-three-in-line problem (see [6,9]). This paper is generalization of [6], where authors analyze the no-three-in-line-problem on the discrete torus. This modified problem is still interesting.

Let m and n be positive integers greater than 1. By a discrete torus $T_{m\times n}$ we mean $\{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, n-1\}$.

Four integers a, b, u, v with $\gcd(u, v) = 1$ correspond to the line $\{(a + uk, b + vk) : k \in \mathbb{Z}\}$ on $\mathbb{Z} \times \mathbb{Z}$. The condition $\gcd(u, v) = 1$ ensures that each pair P, Q of distinct points in $\mathbb{Z} \times \mathbb{Z}$ belongs to exactly one line. For instance, the points $O = (0, 0), P = (2, 2)$ belong to the line $\{(k, k) : k \in \mathbb{Z}\}$.

We define lines on $T_{m\times n}$ to be images of lines in the $\mathbb{Z} \times \mathbb{Z}$ under the projection $\pi_{m,n} : \mathbb{Z} \times \mathbb{Z} \to T_{m\times n}$ defined as follows

$$\pi_{m,n}(a, b) := (a \bmod m, b \bmod n).$$

By x mod y we mean the smallest non-negative remainder when x is divided by y.

We say that a set $X \subset T_{m\times n}$ satisfies the no-three-in-line condition if there are no three collinear points in X. Let $\tau\,(T_{m\times n})$ denote the size of the largest set X satisfying the no-three-in-line condition.

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http://dx.doi.org/10.1016/j.disc.2015.08.006

In our paper we will prove the following theorems.

Theorem 1.1. *We have*

$$\tau(T_{m\times n}) \le 2\gcd(m,n).$$

Theorem 1.2. *We have*

(1) *For* $\gcd(m,n)=1$, $\tau(T_{m\times n})=2$.
(2) *For* $\gcd(m,n)=2$, $\tau(T_{m\times n})=4$.
(3) *Let* $\gcd(m,n)=p$ *be an odd prime.*
  (a) *If* $\gcd(pm,n)=p^2$ *or* $\gcd(m,pn)=p^2$, *then* $\tau(T_{m\times n})=2p$.
  (b) *If* $\gcd(pm,n)=p$ *and* $\gcd(m,pn)=p$, *then* $\tau(T_{m\times n})=p+1$.

Theorem 1.2(1) was proved in [6] by some algebraic argument. Theorem 1.2(2) is a generalized version of Proposition 2.1 from [6]. Similarly, Theorem 1.2(3a) and Theorem 1.2(3b) are generalizations of Theorem 2.7 and Theorem 2.9 from [6], respectively.

2. Proofs of Theorem 1.1 and Theorem 1.2(1)

One of the main tools used in this paper is the Chinese Remainder Theorem.

Theorem 2.1 (*Chinese Remainder Theorem*)**.** *Two simultaneous congruences*

$$x \equiv a \pmod m,\qquad x \equiv b \pmod n$$

are solvable if and only if $a\equiv b \pmod{\gcd(m,n)}$. *Moreover, the solution is unique modulo* $\mathrm{lcm}(m,n)$.

Let us define the family $\mathcal{L}=\{L_s : s\in\{0,1,\dots,\gcd(m,n)-1\}\}$ of lines on $T_{m\times n}$, where

$$L_s=\left\{\pi_{m,n}(k,k-s)\in T_{m\times n} : k\in\mathbb{Z}\right\}.$$

Lemma 2.2 (*See [10]*)**.** *Let* $a=(a_x,a_y)\in T_{m\times n}$ *and* $d=(a_x-a_y) \bmod \gcd(m,n)$. *Then* $a\in L_d$. *Moreover, we have* $L_{s_1}\cap L_{s_2}=\emptyset$ *for* $s_1\ne s_2$ *and* $s_1,s_2\in\{0,1,\dots,\gcd(m,n)-1\}$.

Proof. By Theorem 2.1 there exists $k\in\mathbb{Z}$ such that

$$k\equiv a_x \pmod m,\qquad k\equiv a_y+d \pmod n.$$

Consequently, $(a_x,a_y)=\pi_{m,n}(k,k-d)\in L_d$. Suppose that $L_{s_1}\cap L_{s_2}\ne\emptyset$. This means that there are $k_1,k_2\in\mathbb{Z}$ such that $\pi_{m,n}(k_1,k_1-s_1)=\pi_{m,n}(k_2,k_2-s_2)$. In other words k_1-k_2 is the solution of the following system

$$k_1-k_2\equiv 0 \pmod m,\qquad k_1-k_2\equiv s_1-s_2 \pmod n.$$

By Theorem 2.1 again, we see that $s_1-s_2\equiv 0 \pmod{\gcd(m,n)}$. Hence $L_{s_1}\cap L_{s_2}=\emptyset$ for $s_1\ne s_2$ and $s_1,s_2\in\{0,1,\dots,\gcd(m,n)-1\}$. □

Proof of Theorem 1.1. Let $X\subset T_{m\times n}$ satisfy the no-three-in-line condition. By Lemma 2.2 for every $a\in X$ there exists $L\in\mathcal{L}$ such that $a\in L$. Consequently, $\tau(T_{m\times n})\le 2\cdot|\mathcal{L}|=2\cdot\gcd(m,n)$. □

Proof of Theorem 1.2(1). Obviously it is always true that $\tau(T_{m\times n})\ge 2$. By Theorem 1.1 we get the statement. □

3. Proofs of Theorem 1.2(2) and Theorem 1.2(3a)

Let $X=(x_1,x_2)\in\mathbb{Z}\times\mathbb{Z}$, $Y=(y_1,y_2)\in\mathbb{Z}\times\mathbb{Z}$, $Z=(z_1,z_2)\in\mathbb{Z}\times\mathbb{Z}$. Denote by $D(X,Y,Z)$ the following determinant

$$\begin{vmatrix}1&1&1\\x_1&y_1&z_1\\x_2&y_2&z_2\end{vmatrix}.$$

Recall the determinant criterion for checking whether points are in a line:

Lemma 3.1. *Three points* $X,Y,Z\in\mathbb{Z}\times\mathbb{Z}$ *are in a line if and only if* $D(X,Y,Z)=0$.