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Note

Neighbourhood-width of trees

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ABSTRACT

We determine the relationship between the graph parameters neighbourhood-width and path-width of trees, that turn out equivalent. As our main combinatorial tool, we show that the neighbourhood-width of a tree $T = (V, E)$ is at least $k + 1$ if for some vertex $v \in V$, forest $T[V - \{v\}]$ has at least three non-edgeless components of neighbourhood-width at least k.

1. Introduction

In this paper we consider graph parameters defined by the existence of an underlying path-structure for the input graph: path-width [17], linear clique-width [10], linear NLC-width [10], neighbourhood-width [9], and linear rank-width [8].

Several bounds between these parameters are known and lead to the following relations. A graph has bounded linear clique-width, if and only if it has bounded linear NLC-width, if and only if it has bounded neighbourhood-width, if and only if it has bounded linear rank-width. Only the path-width is less powerful, since a graph of bounded path-width also has bounded neighbourhood-width, but not vice versa. Recursive characterizations are known for several parameters, such as cut-width in [3] and the following one for path-width in [5].

> Let $T = (V, E)$ be a tree and k be some positive integer. The path-width of T is greater than k if and only if there exists $v \in V$ such that $T[V - \{v\}]$ has at least three subtrees all of path-width at least k.

In this paper we introduce a recursive characterization for the neighbourhood-width of trees, which implies a close relationship between the neighbourhood-width and path-width of trees.

For all mentioned parameters the computation problem is NP-hard. For special graph classes there are efficient algorithms for the computation of path-width [20,2] and linear clique-width [11–13]. In this paper we add one result in this direction, since we show that the neighbourhood-width of a forest can be computed in linear time.

2. Preliminaries

We will use standard definitions for graphs which can be found in textbooks, as for example [4]. We denote by P_n the path on n vertices, by K_n the complete graph on n vertices, and by $K_{n,m}$ the complete bipartite graph with n vertices in the one color class and m vertices in the other color class (cf. Fig. 1).

A *k-ary tree* is a rooted tree in which every vertex has at most k children. A k-ary tree is *full*, if every vertex has 0 or k children. A k-ary tree is *perfect*, if it is full and all leaves are on the same level. Some authors call perfect k-ary trees *complete*.

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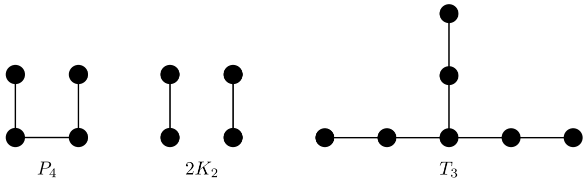

Fig. 1. Special graphs.

A *linear layout* for a graph $G = (V, E)$ is a bijection $\varphi : V \to \{1, \dots, |V|\}$. By $\Phi(G)$ we denote the set of all linear layouts for G. Given $\varphi \in \Phi(G)$ we define for $1 \le i \le |V|$ the left and right sets

$$L(i, \varphi, G) = \{u \in V \mid \varphi(u) \le i\} \quad \text{and} \quad R(i, \varphi, G) = \{u \in V \mid \varphi(u) > i\}.$$

Let $G = (V, E)$ be a graph and $U, W \subseteq V$ be two disjoint vertex sets, by $N_W(u) = \{v \in W \mid \{u, v\} \in E\}$ we denote the restriction of $N_G(u)$ to the vertices in W. By $N(U, W) = \{N_W(u) \mid u \in U\}$ we denote the set of all neighbourhoods of the vertices of set U in set W. This allows us to define the *neighbourhood-width* of graph G, denoted by nw(G), see [9].

$$\mathrm{nw}(G) = \min_{\varphi \in \Phi(G)} \max_{1 \le i \le |V|-1} \Big| N\big(L(i, \varphi, G), R(i, \varphi, G)\big) \Big|.$$

The path-width of a graph G, denoted by pw(G), was defined by Robertson and Seymour in [17] by the existence of a path-decomposition. The path-width of a graph $G = (V, E)$ can also be defined by its *vertex separation number*, denoted by vsn(G), which is defined in [6] as follows.

$$\mathrm{vsn}(G) = \min_{\varphi \in \Phi(G)} \max_{1 \le i \le |V|} \Big| \{u \in L(i, \varphi, G) \mid N_G(u) \cap R(i, \varphi, G) \neq \emptyset\} \Big|.$$

In [14] it was shown, that for every graph the path-width equals its vertex separation number. Using this characterization the following bound for the neighbourhood-width of a graph in its path-width can be shown similar to the linear clique-width bound of Eq. (5) shown in [7].

Lemma 1. *Let G be a graph. Then $nw(G) \le pw(G) + 1$.*

3. A recursive characterization for the neighbourhood-width of trees

Next we show a recursive characterization for the neighbourhood-width of trees.

Theorem 1. *Let $T = (V, E)$ be a tree and v be a vertex of T, such that $T[V - \{v\}]$ has $c \ge 3$ connected components $T_i = (V_i, E_i)$, $1 \le i \le c$. If at least three of the trees T_i have at least one edge and neighbourhood-width at least k, then $nw(T) \ge k + 1$.*

Proof. Let $\varphi \in \Phi(T)$ be a linear layout for T, let T_1, T_2, T_3 be pairwise different connected components of $T[V - \{v\}]$ each having at least one edge, and assume that nw(T_1), nw(T_2), nw$(T_3) \ge k$. There are integers j_1, j_2, j_3 such that, for $i \in \{1, 2, 3\}$

$$\Big| N\big(L(j_i, \varphi, T) \cap V_i, R(j_i, \varphi, T) \cap V_i\big) \Big| \ge k. \tag{1}$$

Let v_1, v_2, v_3 be the vertices of T, such that $\varphi(v_i) = j_i$ for $i \in \{1, 2, 3\}$. We can assume that v_1, v_2, v_3 are vertices of T_1, T_2, T_3, respectively, and $j_1 < j_2 < j_3$.

Let $w_1, w_2, w_3 \in N_T(v)$ where w_i is a vertex of T_i. Define $N(j) := N(L(j, \varphi, T), R(j, \varphi, T))$ for $j \in \{1, \dots, |V|\}$. We first show that $\varphi(w_1) < j_2 < \varphi(v), \varphi(w_3)$ and $\varphi(w_2) \le j_2$.

- If $\varphi(w_3) < j_2$, then $v_3 \neq w_3$ and $|N(j_2)| \ge k + 1$ because of the k neighbourhoods of the vertices of V_2 at j_2 and at least one additional neighbourhood because of the path of at least one edge between w_3 and v_3 in T at j_2, see Fig. 2(a).
- If $j_2 < \varphi(w_1)$, then $v_1 \neq w_1$ and $|N(j_2)| \ge k + 1$ because of the path of at least one edge between v_1 and w_1 in T, see Fig. 2(b).

  Thus we can assume that $\varphi(w_1) < j_2 < \varphi(w_3)$.

- If $\varphi(v) < j_2$, then $|N(j_2)| \ge k + 1$ because of the edge between v and w_3 in T.

  Thus we can assume that $j_2 < \varphi(v)$.

- If $j_2 < \varphi(w_2)$, then $|N(j_2)| \ge k + 1$ because of the edge between v and w_1 in T and w_2 is the only vertex from T_2 adjacent to v.

Thus we can assume that $\varphi(w_2) \le j_2$.

Summarizing all considered cases we now can assume $\varphi(w_1) < j_2 < \varphi(v), \varphi(w_3)$ and $\varphi(w_2) \le j_2$.