Discrete Mathematics 339 (2016) 227–254

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Combinatorial families of multilabelled increasing trees and hook-length formulas

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ARTICLE INFO

Article history:
Received 10 December 2014
Received in revised form 21 July 2015
Accepted 12 August 2015
Available online 5 September 2015

Keywords:
Hook-length formulas
Bilabelled trees
Multilabelled trees
Increasing trees
Elliptic functions
Weierstrass-℘ function
Differential equations

ABSTRACT

In this work we introduce and study various generalizations of the notion of increasingly labelled trees, where the label of a child node is always larger than the label of its parent node, to multilabelled tree families, where the nodes in the tree can get multiple labels.

For all tree classes we show characterizations of suitable generating functions for the tree enumeration sequence via differential equations. Furthermore, for several combinatorial classes of multilabelled increasing tree families we present explicit enumeration results. We also present multilabelled increasing tree families of an elliptic nature, where the exponential generating function can be expressed in terms of the Weierstrass-℘ function or the lemniscate sine function.

Furthermore, we show how to translate enumeration formulas for multilabelled increasing tree families into hook-length formulæ for trees.

Dedicated to Helmut Prodinger on the occasion of his 60th birthday

1. Introduction

1.1. Multilabelled increasing tree families

Increasing trees or *increasingly labelled trees* are rooted labelled trees, where the nodes of a tree T of size $|T| = n$ (where the size $|T|$ of a tree denotes the number of vertices of T) are labelled with distinct integers from a label set $\mathcal{M}$ of size $|\mathcal{M}| = n$ (usually, one chooses as label set the first n positive integers, i.e., $\mathcal{M} = [n] := \{1, 2, \dots, n\}$) in such a way that the label of any node in the tree is smaller than the labels of its children. As a consequence, the labels of each path from the root to an arbitrary node in the tree are forming an increasing sequence, which explains the name of such a labelling.

Several increasing tree models turned out to be appropriate in order to describe the growth behaviour of quantities in various applications and occurred in the probabilistic literature, see [29] for a survey collecting results prior 1995. E.g., they are used to describe the spread of epidemics, to model pyramid schemes, and as a simplified growth model of the world wide web.

First occurrences of increasing trees in the combinatorial literature were due to bijections to other fundamental combinatorial structures, e.g., binary increasing trees and more generally $(d+1)$-ary increasing trees of size n are in bijection to permutations and so-called d-Stirling permutations of order n, respectively, see [19,23,31,36] and references therein. A

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http://dx.doi.org/10.1016/j.disc.2015.08.010

further example is increasingly labelled non-plane unary–binary trees, where it turned out that the number of such trees of size n is twice the number of alternating permutations of order n, see [8,28].

A first systematic treatment of increasing labellings (and related, so-called monotone labellings, where labels are not necessarily distinct) of trees is given by Prodinger and Urbanek in [34], where in particular plane increasing trees (i.e., increasingly labelled ordered trees) of a given size could be enumerated. A fundamental study of increasing tree families yielding exact and asymptotic enumeration results as well as a distributional analysis of various tree parameters is given in [2], see also [17] and references therein.

In above definition of increasing trees each node in the tree gets exactly one label. In this work we introduce and study several extensions of this concept to *multilabelled tree families*, i.e., where the nodes in the tree are equipped with a set or a sequence of labels. Whereas (unilabelled) increasing trees are studied extensively in the combinatorial and probabilistic literature, best to our knowledge the enumeration of increasingly multilabelled trees has not been addressed so far (apart from the author's work [26], where a particular instance appears).

As in [2] we use for our studies a quite general class of weighted trees known as *simple families of trees*, which contains many important combinatorial tree families as particular instances; a formal description is given in Section 2.1. Here we introduce and treat objects obtained when such trees are equipped with certain types of increasing multilabellings as described later. For all multilabelled increasing tree families considered it holds that suitable generating functions for the total weight (i.e., for combinatorial tree families simply the number) of trees of a given size are characterized by certain (higher-order) non-linear differential equations. Interestingly, for several important combinatorial tree models such differential equations either can be solved explicitly or at least can be expressed in terms of special functions as the error function, the Weierstrass-$\wp$ function or the Blasius function, which allow to state explicit enumeration results for trees of a given size.

Besides the enumerative interest in families of increasingly multilabelled trees, we also present so-called *hook-length formulæ* associated to the various tree families. Given a rooted tree T, we call a node $u \in T$ a *descendant* of node $v \in T$ if v is lying on the unique path from the root of T to u. The *hook-length* $h_v := h(v)$ of a node $v \in T$ is defined as the number of descendants of v including the node v itself (i.e., it is the size of the subtree rooted at v).

Various hook-length formulæ for different tree families have been obtained recently, see, e.g., [6,7,12,20,22,32,37]. In particular, Han [22] developed a very versatile expansion technique for deriving hook-length formulæ for partitions and trees. Han's method for trees was extended by Chen et al. [6] and by the authors [27], which allows to determine the "hook-weight function" $\rho(n)$ itself from the considered generating function (or, when considering labelled tree families, the corresponding exponential generating function)

$$G(z) = \sum_{n\geq 1}\Big(\sum_{T\in\mathcal{T}(n)}\prod_{v\in T}\rho(h_v)\Big)z^n,$$

with $\mathcal{T}(n)$ the set of trees of size n of a family $\mathcal{T}$. As a prominent example, Han's expansions technique can be used to give a simple proof of the hook-length formula

$$\sum_{T\in\mathcal{B}(n)}\prod_{v\in T}\left(1+\frac{1}{h_v}\right) = \frac{2^n(n+1)^{n-1}}{n!}$$

for the family of *binary trees* $\mathcal{B}$ obtained by Postnikov [32]. We remark that differential equations in connection with the proof of hook-length formulæ for trees also occur in [20,37]. Furthermore, a remarkable multivariate hook-length formula for unordered increasing trees has been obtained recently by Féray and Goulden [12].

Besides the search and derivation of hook-length formulæ for trees a second important research aspect is to give combinatorial interpretations of them and thus to obtain a "concrete meaning". It is well known (see, e.g., [27] and references therein) that based on the fact that the number of increasing (uni)labellings of a given tree can be described by a nice product formula containing the hook-lengths of the nodes in the tree, enumerative results for (unilabelled) increasing trees can be transferred into hook-length formulæ for corresponding (unlabelled or arbitrary labelled) tree families in a natural way. In this work we extend such considerations to increasing multilabellings of trees and thus are able to provide interpretations of certain hook-length formulaæin terms of families of increasingly multilabelled trees. In particular, we also obtain "elliptic hook-length formulæ": for example, we show that the family $\mathcal{S}$ of so-called *strict-binary trees* satisfies

$$\sum_{T\in\mathcal{S}(n)}\frac{1}{\prod_{v\in T}(2h_v(2h_v-1))} = \frac{(2n+1)2^{3n+4}\pi^{n+1}}{3^{\frac{n-1}{2}}\Gamma^{4n+4}(\frac{1}{4})} \times \sum_{n_1,n_2\in\mathbb{Z}}\frac{1}{(1+n_1+n_2+i(n_1-n_2))^{2n+2}},$$

where $\mathcal{S}(n)$ denotes the number of trees of S of size n and $\Gamma(z)$ the Gamma-function.

It follows an outline of this work, where we give in Table 1 a summary of results concerning interesting particular tree models. In Sections 2–4 we consider bilabelled and more generally k-labelled increasing trees. In *bilabelled increasing trees* or *increasingly bilabelled trees* each node in the tree gets a set of two labels and the labels of a child node are always larger