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Generalized Fibonacci and Lucas cubes arising from powers of paths and cycles

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ABSTRACT

The paper deals with some generalizations of Fibonacci and Lucas sequences, arising from powers of paths and cycles, respectively.

In the first part of the work we provide a formula for the number of edges of the Hasse diagram of the independent sets of the hth power of a path ordered by inclusion. For $h = 1$ such a diagram is called a Fibonacci cube, and for $h > 1$ we obtain a generalization of the Fibonacci cube. Consequently, we derive a generalized notion of Fibonacci sequence, called h-Fibonacci sequence. Then, we show that the number of edges of a generalized Fibonacci cube is obtained by convolution of an h-Fibonacci sequence with itself.

In the second part we consider the case of cycles. We evaluate the number of edges of the Hasse diagram of the independent sets of the hth power of a cycle ordered by inclusion. For $h = 1$ such a diagram is called Lucas cube, and for $h > 1$ we obtain a generalization of the Lucas cube. We derive then a generalized version of the Lucas sequence, called h-Lucas sequence. Finally, we show that the number of edges of a generalized Lucas cube is obtained by an appropriate convolution of an h-Fibonacci sequence with an h-Lucas sequence.

1. Introduction

For a graph $\mathbf{G}$ we denote by $V(\mathbf{G})$ the set of its vertices, and by $E(\mathbf{G})$ the set of its edges.

Definition 1.1. For $n, h \geq 0$,

(i) the *h-power of a path*, denoted by $\mathbf{P}_n^{(h)}$, is a graph with n vertices $v_1, v_2, \ldots, v_n$ such that, for $1 \leq i, j \leq n$, $i \neq j$, $(v_i, v_j) \in E(\mathbf{P}_n^{(h)})$ if and only if $|j - i| \leq h$;

(ii) the *h-power of a cycle*, denoted by $\mathbf{C}_n^{(h)}$, is a graph with n vertices $v_1, v_2, \ldots, v_n$ such that, for $1 \leq i, j \leq n$, $i \neq j$, $(v_i, v_j) \in E(\mathbf{C}_n^{(h)})$ if and only if $|j - i| \leq h$ or $|j - i| \geq n - h$.

Thus, for instance, $\mathbf{P}_n^{(0)}$ and $\mathbf{C}_n^{(0)}$ are the graphs made of n isolated nodes, $\mathbf{P}_n^{(1)}$ is the path with n vertices, and $\mathbf{C}_n^{(1)}$ is the cycle with n vertices. Fig. 1 shows some powers of paths and cycles.

Definition 1.2. An *independent set of a graph* $\mathbf{G}$ is a subset of $V(\mathbf{G})$ not containing adjacent vertices.

Let $\mathbf{H}_n^{(h)}$, and $\mathbf{M}_n^{(h)}$ be the Hasse diagrams of the posets of independent sets of $\mathbf{P}_n^{(h)}$, and $\mathbf{C}_n^{(h)}$, respectively, ordered by inclusion. Clearly, $\mathbf{H}_n^{(0)} \cong \mathbf{M}_n^{(0)}$ is a Boolean lattice with n atoms (n-cube, for short).

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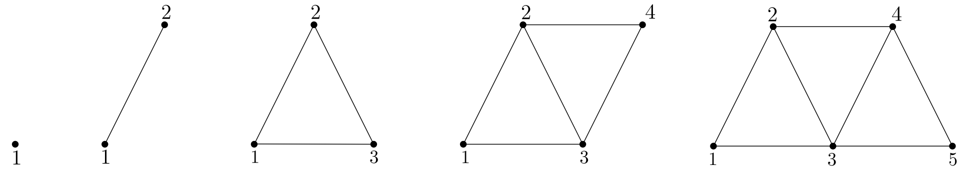
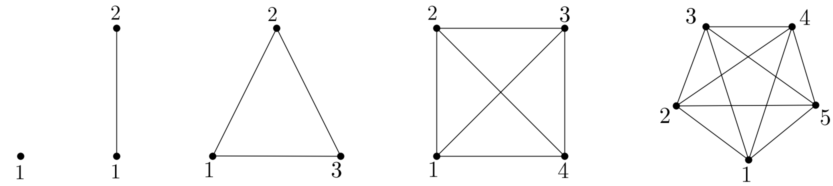


(a) The graphs $\mathbf{P}_1^{(2)}, \ldots, \mathbf{P}_5^{(2)}$.



(b) The graphs $\mathbf{C}_1^{(2)}, \ldots, \mathbf{C}_5^{(2)}$.

Fig. 1. Some powers of paths and cycles.

Before introducing the main results of the paper, we now provide some background on Fibonacci and Lucas cubes. Every independent set S of $\mathbf{P}_n^{(h)}$ can be represented by a binary string $b_1b_2 \cdots b_n$, where, for $i \in \{1, \ldots, n\}$, $b_i = 1$ if and only if $v_i \in S$. Specifically, each independent set of $\mathbf{P}_n^{(h)}$ is associated with a binary string of length n such that the distance between any two 1's of the string is greater than h. Following [10] (see also [6]), a *Fibonacci string of order* n is a binary strings of length n without consecutive 1's. Recalling that the Hamming distance between two binary strings α and β is the number $H(\alpha, \beta)$ of bits where α and β differ, we can define the *Fibonacci cube of order* n, denoted Γ_n, as the graph (V, E), where V is the set of all Fibonacci strings of order n and, for all $\alpha, \beta \in V$, $(\alpha, \beta) \in E$ if and only if $H(\alpha, \beta) = 1$. One can observe that for $h = 1$ the binary strings associated with independent sets of $\mathbf{P}_n^{(h)}$ are *Fibonacci strings of order* n, and the Hasse diagram of the set of all such strings ordered bitwise (*i.e.*, for $S = b_1b_2 \cdots b_n$ and $T = c_1c_2 \cdots c_n$, $S \geq T$ if and only if $b_i \geq c_i$, for every $i \in \{1, \ldots, n\}$) is Γ_n. Fibonacci cubes were introduced as an interconnection scheme for multicomputers in [3], and their combinatorial structure has been further investigated, *e.g.* in [8,10]. Several generalizations of the notion of Fibonacci cubes has been proposed (see, *e.g.*, [4,6]).

Remark. Consider the *generalized Fibonacci cubes* described in [4], *i.e.*, the graphs $\mathbf{B}_n(\alpha)$ obtained from the n-cube $\mathbf{B}_n$ of all binary strings of length n by removing all vertices that contain the binary string α as a substring. In this notation the Fibonacci cube is $\mathbf{B}_n(11)$. It is not difficult to see that $\mathbf{H}_n^{(h)}$ cannot be expressed, in general, in terms of $\mathbf{B}_n(\alpha)$. Instead we have:

$$\mathbf{H}_n^{(2)} = \mathbf{B}_n(11) \cap \mathbf{B}_n(101), \qquad \mathbf{H}_n^{(3)} = \mathbf{B}_n(11) \cap \mathbf{B}_n(101) \cap \mathbf{B}_n(1001), \ldots,$$

where $\mathbf{B}_n(\alpha) \cap \mathbf{B}_n(\beta)$ is the subgraph of $\mathbf{B}_n$ obtained by removing all strings that contain either α or β.

A similar argument can be carried out for cycles. Indeed, every independent set S of $\mathbf{C}_n^{(h)}$ can be represented by a circular binary string (*i.e.*, a sequence of 0's and 1's with the first and last bits considered to be adjacent) $b_1b_2 \cdots b_n$, where, for $i \in \{1, \ldots, n\}$, $b_i = 1$ if and only if $v_i \in S$. Thus, each independent set of $\mathbf{C}_n^{(h)}$ is associated with a circular binary string of length n such that the distance between any two 1's of the string is greater than h. A *Lucas cube of order* n, denoted Λ_n, is defined as the graph whose vertices are the binary strings of length n without either two consecutive 1's or a 1 in the first and in the last position, and in which the vertices are adjacent when their Hamming distance is exactly 1 (see [9]). For $h = 1$ the Hasse diagram of the set of all circular binary strings associated with independent sets of $\mathbf{C}_n^{(h)}$ ordered bitwise is Λ_n. A generalization of the notion of Lucas cubes has been proposed in [5].

Remark. Consider the *generalized Lucas cubes* described in [5], that is, the graphs $\mathbf{B}_n(\widehat{\alpha})$ obtained from the n-cube $\mathbf{B}_n$ of all binary strings of length n by removing all vertices that have a circular containing α as a substring (*i.e.*, such that α is contained in the circular binary strings obtained by connecting first and last bits of the string). In this notation the Lucas cube is $\mathbf{B}_n(\widehat{11})$. It is not difficult to see that $\mathbf{M}_n^{(h)}$ cannot be expressed, in general, in terms of $\mathbf{B}_n(\widehat{\alpha})$. Instead we have:

$$\mathbf{M}_n^{(2)} = \mathbf{B}_n(\widehat{11}) \cap \mathbf{B}_n(\widehat{101}), \qquad \mathbf{M}_n^{(3)} = \mathbf{B}_n(\widehat{11}) \cap \mathbf{B}_n(\widehat{101}) \cap \mathbf{B}_n(\widehat{1001}), \ldots$$

To the best of our knowledge, our $\mathbf{H}_n^{(h)}$, and $\mathbf{M}_n^{(h)}$ are new generalizations of Fibonacci and Lucas cubes, respectively.

In the first part of this paper (which is an extended version of [1]—see remark at the end of this section) we evaluate $p_n^{(h)}$, *i.e.*, the number of *independent sets* of $\mathbf{P}_n^{(h)}$, and $H_n^{(h)}$, *i.e.*, the number of edges of $\mathbf{H}_n^{(h)}$. We then introduce a generalization of the Fibonacci sequence, that we call *h-Fibonacci sequence* and denote by $\mathcal{F}^{(h)}$. Such integer sequence is based on the values