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Note

m-ary partitions with no gaps: A characterization modulo m

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ABSTRACT

In a recent work, the authors provided the first-ever characterization of the values $b_m(n)$ modulo m where $b_m(n)$ is the number of (unrestricted) m-ary partitions of the integer n and $m \geq 2$ is a fixed integer. That characterization proved to be quite elegant and relied only on the base m representation of n. Since then, the authors have been motivated to consider a specific restricted m-ary partition function, namely $c_m(n)$, the number of m-ary partitions of n where there are no "gaps" in the parts. (That is to say, if m^i is a part in a partition counted by $c_m(n)$, and i is a positive integer, then m^{i-1} must also be a part in the partition.) Using tools similar to those utilized in the aforementioned work on $b_m(n)$, we prove the first-ever characterization of $c_m(n)$ modulo m. As with the work related to $b_m(n)$ modulo m, this characterization of $c_m(n)$ modulo m is also based solely on the base m representation of n.

1. Introduction

In this note, we will focus our attention on congruence properties for the partition functions which enumerate restricted integer partitions known as m-ary partitions. These are partitions of an integer n wherein each part is a power of a fixed integer $m \geq 2$. Throughout this note, we will let $b_m(n)$ denote the number of m-ary partitions of n.

As an example, note that there are five 3-ary partitions of $n = 9$:

$$9, \quad 3+3+3, \quad 3+3+1+1+1,$$
$$3+1+1+1+1+1+1, \quad 1+1+1+1+1+1+1+1+1.$$

Thus, $b_3(9) = 5$.

In the late 1960s, Churchhouse [5,6] initiated the study of congruence properties of binary partitions (m-ary partitions with $m = 2$). Within months, other mathematicians proved Churchhouse's conjectures and proved natural extensions of his results. These included Rødseth [9] who extended Churchhouse's results to include the functions $b_p(n)$ where p is any prime as well as Andrews [1] and Gupta [7,8] who proved that corresponding results also held for $b_m(n)$ where m could be any integer greater than 1. As part of an infinite family of results, these authors proved that, for any $m \geq 2$ and any nonnegative integer n, $b_m(m(mn-1)) \equiv 0 \pmod{m}$.

Quite recently, the authors [3] provided the following mod m characterization of $b_m(mn)$ relying solely on the base m representation of n:

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Theorem 1.1. *If $m \geq 2$ is a fixed integer and*

$$n = \alpha_0 + \alpha_1 m + \cdots + \alpha_j m^j$$

is the base m representation of n (so that $0 \leq \alpha_i \leq m-1$ for each i), then

$$b_m(mn) \equiv \prod_{i=0}^{j} (\alpha_i + 1) \pmod{m}.$$

In this note, we provide a similar mod m result for the values $c_m(mn)$, where $c_m(n)$ is the number of m–ary partitions of n with "no gaps" in the parts. More specifically, $c_m(n)$ counts the number of partitions of n into powers of m such that, if m^i is a part in a partition counted by $c_m(n)$, and i is a positive integer, then m^{i-1} must also be a part in the partition. For example, there are six such partitions counted by $c_3(15)$:

$$\begin{aligned}&9+3+1+1+1, \quad 3+3+3+3+1+1+1, \quad 3+3+3+1+1+1+1+1+1,\\ &3+3+1+1+1+1+1+1+1+1+1, \quad 3+1+1+1+1+1+1+1+1+1+1+1+1,\\ &1+1+1+1+1+1+1+1+1+1+1+1+1+1+1.\end{aligned}$$

Note, in particular, that $9+1+1+1+1+1+1$ does not appear in the above list because it does not contain the part 3, and $3+3+3+3+3$ is missing from the list because it does not contain the part 1.

This family of functions $c_m(n)$ is motivated by a recent work of Bessenrodt, Olsson, and Sellers [4] in which the function $c_2(n)$ plays a critical role.

2. The main result

The following theorem provides a complete characterization of $c_m(mn)$ modulo m:

Theorem 2.1. *Let $m \geq 2$ be a fixed integer and let*

$$n = \sum_{i=j}^{\infty} \alpha_i m^i$$

be the base m representation of n where $1 \leq \alpha_j < m$ and $0 \leq \alpha_i < m$ for $i > j$.

(1) *If j is even, then*

$$c_m(mn) \equiv \alpha_j + (\alpha_j - 1) \sum_{i=j+1}^{\infty} \alpha_{j+1} \ldots \alpha_i \pmod{m}.$$

(2) *If j is odd, then*

$$c_m(mn) \equiv 1 - \alpha_j - (\alpha_j - 1) \sum_{i=j+1}^{\infty} \alpha_{j+1} \ldots \alpha_i \pmod{m}.$$

Remark 2.2. Note that Lemma 2.7 (which appears below) implies that Theorem 2.1 tells us the congruence class of $c_m(n)$ modulo m for all n, not just those values of n which are divisible by m.

In order to prove Theorem 2.1, we need a few elementary tools. We describe these tools here.

First, it is important to note the generating function for $c_m(n)$.

Lemma 2.3.

$$C_m(q) := 1 + \sum_{n=0}^{\infty} \frac{q^{1+m+m^2+\cdots+m^n}}{(1-q)(1-q^m)\ldots(1-q^{m^n})}.$$

Proof. The proof follows from a standard argument from [2, Chapter 1]. ■

Next, we wish to find the generating function for $c_m(mn)$.

Lemma 2.4.

$$\sum_{n=0}^{\infty} c_m(mn) q^n = 1 + \frac{q}{1-q} C_m(q) \tag{1}$$