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Dissecting the square into five congruent parts

Liping Yuan [a,b], Carol T. Zamfirescu [c,*], Tudor I. Zamfirescu [d,e,a]

[a] College of Mathematics and Information Science, Hebei Normal University, 050024 Shijiazhuang, PR China
[b] Hebei Key Laboratory of Computational Mathematics and Applications, 050024 Shijiazhuang, PR China
[c] Department of Applied Mathematics, Computer Science and Statistics, Ghent University, Krijgslaan 281 - S9, 9000 Ghent, Belgium
[d] Fachbereich Mathematik, Universität Dortmund, 44221 Dortmund, Germany
[e] "Simion Stoilow" Institute of Mathematics, Roumanian Academy, Bucharest, Roumania

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ABSTRACT

We give an affirmative answer to an old conjecture proposed by Ludwig Danzer: there is a unique dissection of the square into five congruent convex tiles.

1. Introduction and notation

In the eighties of the last century, Ludwig Danzer conjectured in several conferences that there is a unique dissection of the square into five congruent parts—see Fig. 1. In its most general setting, the conjecture asks the parts to be finite unions of closed topological discs.

Danzer formulated his conjecture for the case that the parts are convex, and for the general case as well. We give here an affirmative answer for the case where the parts are convex.

Dissecting convex and other bodies was a frequent occupation of mankind since prehistorical times. We make no attempt here to evoke those efforts and achievements in arts (like painting and cuisine) and sciences, throughout the millennia. As just one example of relatively recent work, we mention Archimedes' "Ostomachion" [1], because he dissected precisely the square.

For many of the mathematical variants, we recommend Grünbaum and Shephard's authoritative book [3], but have to mention the existence of several other important books and surveys in this area.

Danzer's conjecture can be obviously generalized to one in which dissection into n congruent tiles is required, where n is any prime number not less than 3 (see Problem 4). The case $n = 3$ has been solved by Maltby [5].

For points $p, q \in \mathbb{R}^2$, let pq denote the line-segment from p to q, including p and q, and let $|pq|$ be its length. For $M \subset \mathbb{R}^2$, diam M, int$M$, bd$M$, $\mathcal{A}(M)$ denote its diameter, interior, boundary, area, respectively. The convex hull of the finite set $\{a_1, \ldots, a_n\} \subset \mathbb{R}^2$ will be denoted by $a_1 \ldots a_n$. The circle with centre x and radius r will be denoted by $C(x, r)$.

Consider the square $Q = [0, 1]^2$.

A compact convex set $K \subset \mathbb{R}^2$ is called here a *tile*, if Q is the union of five congruent copies of K such that any two of them are either disjoint or have just boundary points in common. Throughout the paper, these five tiles will be denoted

* Corresponding author.
E-mail addresses: lpyuan@mail.hebtu.edu.cn (L. Yuan), czamfirescu@gmail.com (C.T. Zamfirescu), tudor.zamfirescu@mathematik.uni-dortmund.de (T.I. Zamfirescu).

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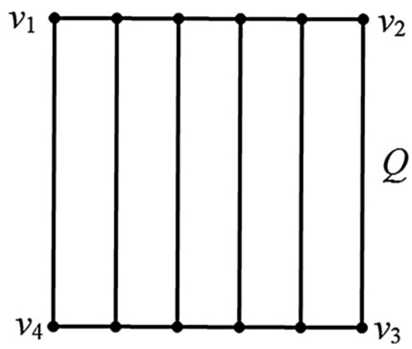

Fig. 1.

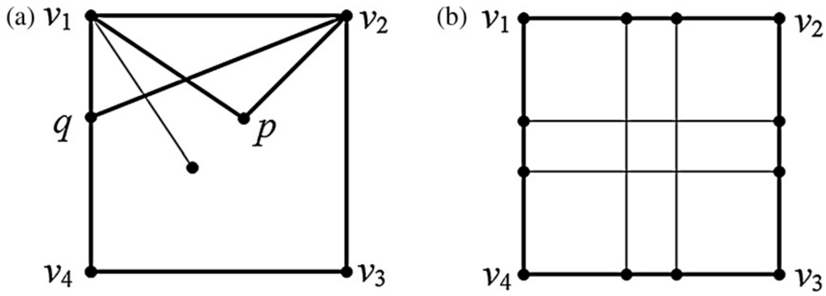

Fig. 2.

by $K_1, \ldots, K_5$. Obviously, K must be a convex polygon. Indeed, since the convex tiles K_i form a tiling of the square, the intersection of two tiles is either empty, or a single point, or a line-segment; so K has a boundary consisting of finitely many line-segments, and hence is a polygon. We will call here this particular dissection a *tiling*.

The boundaries of the five tiles form a graph, which has as vertices the vertices of the tiles and as edges their sides or parts of them, joining those vertices. We use the same term of *tiling* when referring to this graph.

Let $v_1 = (0, 1)$, $v_2 = (1, 1)$, $v_3 = (1, 0)$, $v_4 = (0, 0)$, and put $v_5 = v_1$. So Q has vertices v_i and sides $v_i v_{i+1}$ ($i = 1, \ldots, 4$). Put $Q^* = \mathrm{bd}Q \setminus \{v_1, v_2, v_3, v_4\}$.

The main steps of our proof of Danzer's conjecture are these: first, we eliminate the possibility that the tiles are triangles. Then we eliminate several topologically different cases of tiling the square Q. Third, we show that some edge of Q must contain no vertex of the tiling, which provides the strong geometric property of the tiles of having a side of length 1. Finally, we are led to the obvious tiling.

2. Preparation

Lemma 1. *K is not a triangle.*

Proof. This is a direct consequence of Monsky's theorem saying that there is no tiling of the square into an odd number of triangles of equal areas [6]. Although the proof of Monsky's theorem is elegant and not too long, we give here a very simple argument for (the weaker) Lemma 1.

Suppose there exists a tiling of Q into five congruent triangles. The angle sum of the five triangles is 5π. The sum of the angles in the four corners of Q is 2π. Therefore, further vertices must account for precisely 3π—thus, they are at least two and at most three. Choose the points $p = (3/5, 3/5)$, $q = (0, 3/5)$.

Claim 1. The triangle $v_1 v_2 p$ cannot be a tile. Indeed, suppose it is. We have $\angle v_1 p v_2 > \pi/2$, $\angle p v_2 v_1 = \pi/4$, $\angle v_2 v_1 p = \arctan \frac{2}{3}$.

Then another tile must have a vertex at v_1. Its angle there can only measure $\pi/4$ or $\arctan \frac{2}{3}$. Therefore there must be a further tile with vertex at v_1. The remaining angle at v_1 for this tile is at most

$$\frac{\pi}{2} - 2\arctan\frac{2}{3} < \arctan\frac{2}{3},$$

so this is impossible. See Fig. 2(a).

Claim 2. The triangle $v_1 v_2 q$ cannot be a tile. Indeed, suppose it is. We have $|qv_1| = 2/5$, $|v_1 v_2| = 1$, $|v_2 q| = \sqrt{29}/5$. Then $|qv_4| = 3/5$. As a line-segment of length $3/5$ is not a union of line-segments of length at least $2/5$ with pairwise disjoint relative interiors, Claim 2 is true.