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The 3-colorability of planar graphs without cycles of length 4, 6 and 9

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ABSTRACT

In this paper, we prove that planar graphs without cycles of length 4, 6, 9 are 3-colorable.

1. Introduction

The well-known Four Color Theorem states that every planar graph is 4-colorable. On the 3-colorability of planar graphs, a famous theorem owing to Grötzsch [6] states that every planar graph without cycles of length 3 is 3-colorable. Therefore, next sufficient conditions that guarantee 3-colorability of planar graphs should always allow the presence of cycles of length 3. In 1976, Steinberg conjectured that every planar graph without cycles of length 4 and 5 is 3-colorable. Erdös [9] suggested a relaxation of Steinberg's Conjecture: does there exist a constant k such that every planar graph without cycles of length from 4 to k is 3-colorable? Abbott and Zhou [1] proved that such a constant exists and $k \le 11$. This result was later on improved to $k \le 9$ by Borodin [2] and, independently, by Sanders and Zhao [8], and to $k \le 7$ by Borodin, Glebov, Raspaud and Salavatipour [5]. Besides, much attention was paid to sufficient conditions that forbid cycles of some other certain lengths. The results concerning four kinds of forbidden lengths of cycles were obtained in several different papers and summarized in [7]:

Theorem 1.1. *If G is a planar graph that has no cycles of length $4, i, j, k$, where $5 \le i < j < k \le 9$, then G is 3-colorable.*

A more general problem than Steinberg's was formulated also in [7]:

Problem 1.2. What is $\mathcal{A}$, a set of integers between 5 and 9, such that for $i \in \mathcal{A}$, every planar graph with cycles of length neither 4 nor i is 3-colorable?

It seems very far to settle Problem 1.2, since no element of such a set $\mathcal{A}$ has been found. Therefore, a reasonable way to deal with this problem is to ask following question:

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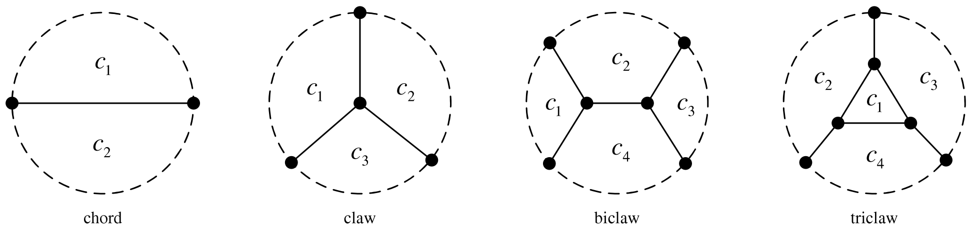

Fig. 1. Chord, claw, biclaw and triclaw of a cycle.

Problem 1.3. What is $\mathcal{B}$, a set of pairs of integers (i, j) with $5 \le i < j \le 9$, such that planar graphs without cycles of length $4, i, j$ are 3-colorable?

The first step towards Problem 1.3 was made by Xu [11], who proved that a planar graph is 3-colorable if it has neither 5- and 7-cycles nor adjacent 3-cycles. Unfortunately, there is a gap in his proof, as pointed out by Borodin et al. [3], who later on gave a new proof of the same statement. Afterwards, Xu [12] fixed this gap. Hence $(5, 7) \in \mathcal{B}$. Other known elements of $\mathcal{B}$ includes the pair $(6, 8)$ given by Wang and Chen [10], the pair $(7, 9)$ given by Lu et al. [7], and the pair $(6, 7)$ given by Borodin, Glebov and Raspaud [4]. Actually, the theorem proved in [4] states that planar graphs without triangles adjacent to cycles of length from 4 to 7 are 3-colorable, which implies $(6, 7) \in \mathcal{B}$.

In this paper, we show $(6, 9) \in \mathcal{B}$, that is, we prove the following theorem:

Theorem 1.4. *Every planar graph without cycles of length* 4, 6, 9 *is* 3-*colorable.*

The graphs considered in this paper are finite and simple. Let G be a plane graph and C a cycle of G. By $Int(C)$ (or $Ext(C)$) we denote the subgraph of G induced by the vertices lying inside (or outside) of C. Cycle C is *separating* if both $Int(C)$ and $Ext(C)$ are not empty. By $\overline{Int}(C)$ (or $\overline{Ext}(C)$) we denote the subgraph of G consisting of C and its interior (or exterior).

Denote by $G[S]$ the subgraph of G induced by S, where either $S \subseteq V(G)$ or $S \subseteq E(G)$. A vertex is a *neighbor* of another vertex if they are adjacent. A *chord* of C is an edge of $\overline{Int}(C)$ that connects two nonconsecutive vertices on C. If $Int(C)$ has a vertex v with three neighbors v_1, v_2, v_3 on C, then $G[\{vv_1, vv_2, vv_3\}]$ is called a *claw* of C. If $Int(C)$ has two adjacent vertices u and v, both of them have two neighbors on C, say u_1 and u_2 adjacent to u, and v_1 and v_2 adjacent to v, then $G[\{uv, uu_1, uu_2, vv_1, vv_2\}]$ is called a *biclaw* of C. If $Int(C)$ has three pairwise adjacent vertices u, v, w, each of them has a neighbor on C, say u', v', w' respectively, then $G[\{uv, vw, uw, uu', vv', ww'\}]$ is called a *triclaw* of C (see Fig. 1).

Let C be a cycle and T be one of the chords, claws, biclaws and triclaws of C. We call the graph H consisting of C and T a *bad partition* of C. The boundary of any one of the parts, into which C is divided by H, is called a *cell* of H. Clearly, every cell is a cycle. In case of confusion, let us always order the cells $c_1, \dots, c_t$ of H in the way as shown in Fig. 1. For each cell c_i of H, let k_i be the length of c_i. T is further called a (k_1, k_2)-chord, a (k_1, k_2, k_3)-claw, a (k_1, k_2, k_3, k_4)-biclaw or a (k_1, k_2, k_3, k_4)-triclaw, respectively.

Let k be a positive integer. A k-cycle is a cycle of length k. A k^--cycle (or k^+-cycle) is a cycle of length at most (or at least) k. A *good cycle* is a 12^--cycle that has none of claws, biclaws and triclaws. A *bad cycle* is a 12^--cycle that is not good. We say a 9-cycle is *special* if it has a $(3, 8)$-chord or a $(5, 5, 5)$-claw.

Let $\mathcal{G}$ be the class of connected plane graphs with neither 4- and 6-cycle nor special 9-cycle.

Instead of Theorem 1.4, it is easier for us to prove the following stronger one:

Theorem 1.5. *Let $G \in \mathcal{G}$. We have*

(1) *G is* 3-*colorable; and*
(2) *if D, the boundary of the exterior face of G, is a good cycle, then every proper* 3-*coloring of* $G[V(D)]$ *can be extended to a proper* 3-*coloring of G.*

This section is concluded with some other notations that are used in the next section. Let G be a plane graph. Denote by $d(v)$ the degree of a vertex v, by $|C|$ the length of a cycle C and by $|f|$ the size of a face f. Let k be a positive integer. A *k-vertex* is a vertex of degree k, and a *k-face* is a face of size k. A *k^+-vertex* (or *k^--vertex*) is a vertex of degree at least (or at most) k, and similarly, a *k^+-face* (or *k^--face*) is a face of size at least (or at most) k. A *k-path* is a path that contains k edges. A k-cycle containing vertices $v_1, \dots, v_k$ in cyclic order is denoted by $[v_1 \dots v_k]$. Denote by $N(v)$ the set of neighbors of a vertex v. Let $N_H(v) = N(v) \cap V(H)$ whenever v is a vertex of a subgraph H of G. A vertex is *external* if it lies on the exterior face; *internal* otherwise. A vertex incident with a triangle is called a *triangular vertex*. We say a vertex is *bad* if it is an internal triangular 3-vertex; *good* otherwise. A path is a *splitting path* of a cycle C if it has two end-vertices on C and all other vertices inside C. We say a path is *good* if it is of length 3, contains only internal 3-vertices and has an end-edge incident with a triangle. A cycle or a face C is *triangular* if C is adjacent to a triangle T. Furthermore, if C is a cycle and $T \in \overline{Ext}(C)$, then we say C is an *ext-triangular* cycle. A triangular 7-face is *light* if it has no external vertex and every incident nontriangular vertex has degree 3.