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Colorings and spectral radius of digraphs

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ABSTRACT

We determine the digraphs that have the minimum and second minimum spectral radius among all strongly connected digraphs with given order and dichromatic number. This solves a problem posed by Lin, Shu, Wu, and Yu.

1. Introduction

Throughout this article, we consider finite, simple strongly connected digraphs, i.e. without loops and multiple arcs. We use standard terminology and notation and refer to [1] for an extensive treatment of digraphs. For a digraph $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ are the vertex set and arc set of G, respectively. If $e = \overrightarrow{\alpha\beta} \in E(G)$, then α is the initial vertex of e and β is the terminal vertex. The *outdegree* $d_G^+(\gamma)$ of a vertex γ is the number of arcs of which it is the initial vertex and the *indegree* $d_G^-(\gamma)$ is the number of arcs of which it is the terminal vertex. We call G *strongly connected* if whenever $\alpha, \beta \in V(G)$, there exists a directed path from α to β.

The complete digraph $\overrightarrow{K_k}$ of order k is the digraph in which every ordered pair of distinct vertices defines an arc. The digraph $B_{n,k}$ is the digraph of order n obtained by adding a directed path from one vertex of $\overrightarrow{K_k}$ to *another* vertex of $\overrightarrow{K_k}$. In this process, $n-k$ vertices are added and $n-k+1$ arcs. We denote by $Q_{n,k}$ the digraph of order n obtained by adding a directed path from one vertex of $\overrightarrow{K_k}$ to the *same* vertex of $\overrightarrow{K_k}$. We also denote by $\overrightarrow{C_n}$ the directed cycle of length n and $\overleftrightarrow{C_n}$ the bidirected cycle of length n.

Let G be a digraph. A vertex set $A \subseteq V(G)$ is *acyclic* if the induced subdigraph $G[A]$ is acyclic. A partition of $V(G)$ into k acyclic sets is called a k-coloring of G. The minimum integer k for which there exists a k-coloring of G is the dichromatic number $\chi(G)$ of the digraph G. This definition was introduced by Neumann-Lara [8] and extends the concept of the chromatic number of a graph in a natural way.

If G is a digraph and α is a vertex of G such that the removal of α from G along with all the arcs involving α yields a digraph with strictly smaller dichromatic number, then we say that α is a *critical vertex*. If $\chi(G) = k$ and every vertex of G is critical, we say that G is a *k-vertex critical* digraph. The stronger concept of a *k-arc critical* digraph is not relevant to this article. Mohar [7, Lemma 2.2 and Theorem 2.3] states that if G is a k-vertex critical digraph then every indegree and outdegree is $\geq k-1$. Further if the digraph is regular of degree $k-1$ then one of the following cases occurs

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- $k = 2$ and $G \cong \overrightarrow{C_n}$ with $n \geq 2$,
- $k = 3$ and $G \cong \overleftrightarrow{C_n}$ with n odd, $n \geq 3$,
- $G \cong \overrightarrow{K_k}$.

For a digraph G we denote by $A(G)$ the adjacency matrix of G. For any square matrix M with complex entries, the spectral radius $\rho(M)$ of M is defined by

$$\rho(M) = \max\{|\lambda|; \lambda \text{ is an eigenvalue of } M\}.$$

It is well known that if M has nonnegative entries, then $\rho(M)$ is an eigenvalue of M. If G is a strongly connected digraph, then $A(G)$ is an irreducible matrix with nonnegative entries and $\rho(A(G))$ is strictly positive and is a simple eigenvalue of $A(G)$. We refer the reader to [4, chapter 8] for information about nonnegative matrices. We will denote by $\mathbb{1}$ the vector in which every entry is 1 and by $\mathbb{1}_X$ the vector in which the entries in X are 1 and the others are 0.

In [5], the authors defined θ-digraph as follows. The θ-digraph consists of three directed paths P_{a+2}, P_{b+2} and P_{c+2} such that the initial vertex of P_{a+2} and P_{b+2} is the terminal vertex of P_{c+2}, and the initial vertex of P_{c+2} is the terminal vertex of P_{a+2} and P_{b+2}, denoted by $\theta(a, b, c)$. In the same paper, the authors ask if $\theta(0, 1, n-3)$ attains the second smallest spectral radius among all strongly connected digraphs. Hong and You [3] answered this question affirmatively. Clearly, $\chi(\theta(0, 1, n-3)) = 2$.

The objective of this article is to establish the following theorem, thus solving the problem of Lin et al. [6, Problem 1].

Theorem 1.1. • *Let $k \geq 4$. Then among all strongly connected digraphs G of order n and $\chi(G) = k$, the unique digraph that minimizes the $\rho(A(G))$ is the digraph $B_{n,k}$. The unique second minimal such digraph is $Q_{n,k}$.*
- *If $n \geq 3$ is odd, then the minimal strongly connected digraph of order n and $\chi(G) = 3$ is the $\overleftrightarrow{C_n}$ and the second minimal such digraph is $B_{n,3}$.*
- *If $n \geq 4$ is even, then the minimal strongly connected digraph of order n and $\chi(G) = 3$ is $B_{n,3}$ and the second minimal such digraph is $Q_{n,3}$.*
- *Among all strongly connected digraphs G of order $n \geq 3$ and $\chi(G) = 2$ the unique digraph that minimizes the $\rho(A(G))$ is the digraph $\overrightarrow{C_n}$. The unique second minimal such digraph is $\theta(0, 1, n-3)$.*

We remark that the corresponding theorem for graphs was solved in Feng et al. [2]. A proof of their theorem can also be found in [9, page 94].

2. Estimating the Perron root of $A(B_{n,k})$ and $A(Q_{n,k})$

Lemma 2.1. *For $2 \leq k < n$ the characteristic polynomial of the adjacency matrix of $B_{n,k}$ is given by*

$$\det(\lambda I - A(B_{n,k})) = (\lambda+1)^{k-2}\Big(\lambda^{n+2-k} - (k-2)\lambda^{n+1-k} - (k-1)\lambda^{n-k} - 1\Big).$$

Proof. We denote by J the $k \times k$ matrix of all ones, by N the $(n-k) \times (n-k)$ matrix with ones on the superdiagonal and zeros elsewhere, by C_1, C_2 and C_3 the matrices of shapes $k \times (n-k)$, $(n-k) \times k$ and $(n-k) \times (n-k)$ respectively with a one at the bottom left-hand corner and zeros elsewhere. Then we may write with suitable labeling of the vertices of $B_{n,k}$

$$\lambda I - A(B_{n,k}) = \begin{pmatrix} (\lambda+1)I - J & -C_1 \\ -C_2 & \lambda I - N \end{pmatrix}$$

where I denotes the identity matrix of appropriate size. Then, using a well-known Schur complements formula we have for $\alpha = (\lambda+1)^{-1}, \beta = (\lambda+1)^{-1}(\lambda+1-k)^{-1}$

$$\begin{aligned}
\det(\lambda I - A(B_{n,k})) &= \det(\lambda I - N)\det((\lambda+1)I - J)\det(I - C_2((\lambda+1)I - J)^{-1}C_1(\lambda I - N)^{-1}) \\
&= \lambda^{n-k}(\lambda+1)^{k-1}(\lambda+1-k)\det(I - C_2(\alpha I + \beta J)C_1(\lambda I - N)^{-1}) \\
&= \lambda^{n-k}(\lambda+1)^{k-1}(\lambda+1-k)\det(I - \beta C_2 J C_1(\lambda I - N)^{-1}) \\
&= \lambda^{n-k}(\lambda+1)^{k-1}(\lambda+1-k)\det(I - \beta C_3(\lambda I - N)^{-1}) \\
&= \lambda^{n-k}(\lambda+1)^{k-1}(\lambda+1-k)\Big(1 - \beta \operatorname{tr}(C_3(\lambda I - N)^{-1})\Big) \\
&= (\lambda+1)^{k-2}\Big(\lambda^{n+2-k} - (k-1)\lambda^{n+1-k} - (k-1)\lambda^{n-k} - 1\Big)
\end{aligned}$$

using at various points $C_2C_1 = 0$ for $k \geq 2$, $C_2JC_1 = C_3$, $\det(I - X) = 1 - \operatorname{tr}(X)$ if X has rank one and by an easy matrix calculation $\operatorname{tr}(C_3(\lambda I - N)^{-1}) = \lambda^{-(n-k)}$. $\square$

Lemma 2.2. *For $2 \leq k < n$ the characteristic polynomial of the adjacency matrix of $Q_{n,k}$ is given by*

$$\det(\lambda I - A(Q_{n,k})) = (\lambda+1)^{k-2}\Big(\lambda^{n+2-k} - (k-2)\lambda^{n+1-k} - (k-1)\lambda^{n-k} - \lambda + (k-2)\Big).$$