Discrete Mathematics 339 (2016) 333–343

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Improved bounds on the partitioning of the Boolean lattice into chains of equal size

István Tomon

University of Cambridge, United Kingdom

ARTICLE INFO

Article history:
Received 8 March 2014
Received in revised form 28 August 2015
Accepted 31 August 2015
Available online 12 September 2015

Keywords:
06A07
Boolean lattice
Chain partition
Power of poset
Conjecture of Füredi
Conjecture of Griggs

ABSTRACT

The Boolean lattice $2^{[n]}$ is the power set of $[n]$ ordered by inclusion. If c is a positive integer, a c-partition of a poset is a chain partition, where all but at most one of the chains have size c. We prove that if $n = \Omega(c^2)$, then $2^{[n]}$ has a c-partition. This improves a theorem of Lonc.

We also prove a generalization of this result. If c is a positive integer and P is a poset whose comparability graph is connected, then P^n has a c-partition if n is sufficiently large.

1. Introduction

The Boolean lattice $2^{[n]}$ is the power set of $[n]$ ordered by inclusion, where $[n] = \{1, \ldots, n\}$. A *chain* in a poset is a subset C, whose elements can be named $c_1, c_2, \ldots, c_r$ such that $c_1 < c_2 < \cdots < c_r$.

Let c be a positive integer. A partition of a poset into chains $C_1, \ldots, C_s$ is called a *c-partition*, if all but at most one of the chains $C_1, \ldots, C_s$ has size c. If $|C_i| \neq c$, call C_i the *exceptional chain*.

Sands [11] conjectured that for any positive integer m there exists N such that if $n \geq N$ then $2^{[n]}$ can be partitioned into chains of size 2^m. Let $N(m)$ denote the least such N. It is trivial that $N(1) = 1$; Griggs, Yeh and Grinstead [6] proved that $N(2) = 9$.

In 1988 Griggs [5] also proposed a stronger conjecture: for any positive integer c there exists N' such that if $n \geq N'$, then $2^{[n]}$ has a c-partition. Let $N'(c)$ denote the least such N'. Lonc [9] showed that the conjecture is indeed true, and in [2] it was established that $N'(c) < 2^{2^{36c^2}}$.

In this paper, we shall improve the bounds on N and N'. We prove that $N'(c) = O(c^2)$, which is sharp up to a constant. Our paper is organized as follows.

In Section 2, we shall give a short proof that $N(m) = O(2^{2m}m)$ and present the main ideas used throughout this paper. We shall improve these ideas in the following sections to prove that $N(m) = O(2^{2m})$, and in general $N'(c) = O(c^2)$.

We also prove the following generalization of the theorem of Lonc. We show that for any positive integer c and poset P whose comparability graph is connected, there exists $N(P, c)$ such that if $n \geq N(P, c)$, then the cartesian power P^n has a c-partition. We first prove this for $c = |P|$ and the c-partition having no exceptional chain. Later, we extend our ideas to prove the general case.

E-mail address: it262@cam.ac.uk.

http://dx.doi.org/10.1016/j.disc.2015.08.025
0012-365X/

Before we proceed with our proofs, let us mention some of the main results we shall use throughout this paper. As we shall see, one of the main obstacles in finding a c-partition is that it is not easy to construct a chain partition, where each chain is "large". One of the main conjectures addressing this problem is the following conjecture of Füredi [3].

Conjecture 1.1. *The Boolean lattice $2^{[n]}$ can be partitioned into $\binom{n}{\lfloor n/2\rfloor}$ chains such that the sizes of the chains differ in at most 1.*

This conjecture is a special case of the following conjecture of Griggs [5].

Conjecture 1.2. *Let n be a positive integer and $m = \binom{n}{\lfloor n/2\rfloor}$. Let*

$$\mu = (\mu_1, \dots, \mu_m)$$

be a partition of 2^n into positive integers, where $\mu_1 \geq \cdots \geq \mu_m$ and let

$$\sigma = (\sigma_1, \dots, \sigma_m)$$

be the partition of 2^n, where $\sigma_i = n - 2l + 1$, if $\binom{n}{l-1} < i \leq \binom{n}{l+1}$ with some $0 \leq l \leq \lfloor n/2\rfloor$. If

$$\sum_{i=1}^{j} \mu_i \leq \sum_{i=1}^{j} \sigma_i$$

holds for $j = 1, \dots, m$, then $2^{[n]}$ can be partitioned into m chains such that the sizes of the chains are $\mu_1, \dots, \mu_m$, respectively.

Both conjectures are still open. If either Griggs's or Füredi's conjecture is true, $2^{[n]}$ has a partition into chains of size $\sim\sqrt{\pi/2}\sqrt{n}$. However, there are some partial results towards Conjecture 1.2. Hsu et al. [7,8] proved that $2^{[n]}$ can be partitioned into $\binom{n}{\lceil n/2\rceil}$ chains of size between $\sqrt{n}/2 + O(1)$ and $c\sqrt{n}\log n$, with some absolute constant c. The author of this paper [12] proved the following improvement on these bounds.

Theorem 1.3 ([12]). *If n is sufficiently large, the Boolean lattice $2^{[n]}$ can be partitioned into $\binom{n}{\lfloor n/2\rfloor}$ chains such that the size of each chain is between $0.8\sqrt{n}$ and $13\sqrt{n}$.*

2. A proof of the conjecture of Sands

In this section, we give a short proof of the conjecture of Sands. This proof gives the bound $N(m) = O(m2^{2m})$, which we shall improve in the third section to eliminate the factor m.

Theorem 2.1. *Let m be a positive integer. If $n = \Omega(m2^{2m})$, then $2^{[n]}$ can be partitioned into chains of size 2^m.*

Before we embark on our proof, let us recall some basic definitions and notations regarding posets.

The relation in a poset P is denoted by $<_P$ or simply $<$ if it is clear from the context which poset is under consideration.

We shall regard the *cartesian product* of the posets P and Q as a poset on the set $P \times Q$ in which $(p_1, q_1) \leq (p_2, q_2)$ if and only if $p_1 \leq p_2$ and $q_1 \leq q_2$. For a positive integer n we shall denote by P^n the poset $P \times \cdots \times P$, where the cartesian product contains n factors.

If $k_1, \dots, k_d$ are positive integers, then

$$[k_1] \times \cdots \times [k_d]$$

is the d-dimensional *grid*. Clearly, a grid is a poset with the usual pointwise order. That is $(a_1, \dots, a_k) \leq (b_1, \dots, b_k)$ if and only if $a_1 \leq b_1, \dots, a_k \leq b_k$. Grids with $k = 2$ will be sometimes called *rectangles*. In fact, we also call posets isomorphic to $[k_1] \times \cdots \times [k_d]$ grids. In this paper, we shall often view $2^{[n]}$ as the grid $[2]^n$.

We shall use the following two simple lemmas throughout the paper, which we state without proof.

Lemma 2.2. *Let P and Q be posets and c a positive integer.*

(i) *If P has a partition into chains, each of size c, then $P \times Q$ has a partition into chains, each of size c.*
(ii) *If P has a partition into chains, each of size at least c, then $P \times Q$ has a partition into chains, each of size at least c.*

Lemma 2.3. *Let $P_1, \dots, P_k$ be posets, and for $i \in [k]$ let $\{Q_{i,j}\}_{j=1}^{s_i}$ be a partition of P_i. The family*

$$\{Q_{1,j_1} \times \cdots \times Q_{k,j_k} : 1 \leq j_i \leq s_i,\ \text{for } i \in [k]\}$$

is a partition of $P_1 \times \cdots \times P_k$.

It turns out that the analogue of Griggs's conjecture [5] for grids of large sides is easier to prove. We shall make use of this observation to deduce our theorem for the Boolean lattice in the following way. Instead of partitioning $2^{[n]}$ into chains, we partition it to appropriate grids of large sides. Then we further partition those grids into chains.