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A cut locus for finite graphs and the farthest point mapping

Alessandro Maddaloni [a,*], Carol T. Zamfirescu [b]

[a] Department of Mathematics & Computer Science, University of Southern Denmark, Denmark

[b] Department of Applied Mathematics, Computer Science and Statistics, Ghent University, Krijgslaan 281-S9, 9000 Ghent, Belgium

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ABSTRACT

We reflect upon an analogue of the cut locus, a notion classically studied in Differential Geometry, for finite graphs. The *cut locus* $C(x)$ of a vertex x shall be the graph induced by the set of all vertices y with the property that no shortest path between x and z, $z \neq y$, contains y. The cut locus coincides with the graph induced by the vertices realizing the local maxima of the distance function. The function F mapping a vertex x to $F(x)$, the set of global maxima of the distance function from x, is the *farthest point mapping*. Among other things, we observe that if, for a vertex x, $C(x)$ is connected, then $C(x)$ is the graph induced by $F(x)$, and prove that the farthest point mapping has period 2. Elaborating on the analogy with Geometry, we study graphs satisfying *Steinhaus' condition*, i.e. graphs for which the farthest point mapping is single-valued and involutive.

1. Introduction

The main goal of this paper is to study the class of graphs for which the function which maps a vertex x to the set of vertices farthest from x (with respect to the metric defined via shortest paths), the so-called *farthest point mapping*, is both single-valued and involutive. (Note that in more established graph-theoretical terms, the farthest vertices from a fixed vertex x are just the vertices whose distance to x coincides with the eccentricity of x.) For an overview of results concerning distance in graphs, see the monograph by Buckley and Harary [9], and the surveys by Goddard and Oellermann [19], and Chartrand and Zhang [14]. The following two aspects will be reflected upon:

(i) We will work with the notion of "cut locus" from Differential Geometry (see [31], and for a "panoramic" view [4]; its connection to the famous optimal transport problem can be found in [38]), and analyse consequences of its application in Graph Theory. The geometric origins of the cut locus, in their most general setting, lie in a metric space (X, ρ), the metric of which is intrinsic. This motivates the graph theoretical definition, which mimics – as far as possible – its geometric analogue. Consider $x \in X$. $y \in X$ is a *cut point* of x if for any segment γ from x to y there exists no other segment $\sigma \supsetneq \gamma$ starting at x. The set of all cut points of x is called the *cut locus* of x. As it is of central importance, let us emphasize what we understand by *segment*: a shortest path between x and y, i.e. an arc of length $\rho(x, y)$. Explicitly, the geometric concept of the cut locus was introduced by Poincaré [36] in 1905, who used the term "ligne de partage", whereas implicit use occurred at least as early as 1881, in a paper by von Mangoldt [39]. Poincaré's work on surfaces was extended to Riemannian manifolds in the 1930s by Whitehead [40] (who coined the term "cut locus") and Myers [32,33].

(ii) We also investigate graphs for which the farthest point mapping need not be single-valued, but the set of vertices realizing the local maxima and the set of vertices realizing the global maxima of the distance function coincide for every vertex of the graph. The underlying motivation for studying these graphs is a natural consequence of the involutivity of the

* Corresponding author.
E-mail addresses: maddaloni@imada.sdu.dk (A. Maddaloni), czamfirescu@gmail.com (C.T. Zamfirescu).

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farthest point mapping, the results arising from (i), and the study of the *injectivity radius* (i.e. the minimum over all distances between a vertex and its cut locus), another concept borrowed from Differential Geometry and relevant to our pursuits.

Recently, O'Rourke asked for translating the idea of the cut locus to Graph Theory [34]. Itoh and Vîlcu also connected the concepts of cut locus and graphs, but in a different sense: One of their main results is that for every graph G there exists a surface S and a point in S whose cut locus is isomorphic to G; rephrasing, every graph can be realized as a cut locus. This, and related results, can be found in a series of four papers [23–26].

1.1. Preliminaries

Throughout this paper all graphs will be simple, i.e. without loops or multiple edges[1], undirected, and finite, and have order at least 3, unless explicitly mentioned otherwise. For the inclusion relation between sets we will use the following notation. The symbol $\subset$ shall designate inclusion where equality may hold, and $\subsetneq$ inclusion where equality is forbidden.

Let $G = (V(G), E(G))$ be a connected graph, where $V(G)$ ($E(G)$) denotes its vertex (edge) set. For a set S we will denote by $|S|$ its cardinality. The edge between two vertices u and v will be denoted by uv. For vertex sets or subgraphs X, Y, put $E(X, Y) = \{xy : x \in X, y \in Y\}$. For a set of vertices or a (possibly disconnected) subgraph $H \subset G$, we shall use $\langle H\rangle$ for the graph induced by H. We also need the (set-theoretical) *complement* $\complement H = \langle V(G - H)\rangle$. We say that H *dominates* G if every vertex of $V(G) - H$ is adjacent with a vertex of H. Abusing notation, we will make no distinction between a vertex x, the graph $(\{x\}, \emptyset) \cong K_1$ and the set $\{x\}$. For two graphs G and G' we denote by $G\square G'$ the Cartesian product of G and G'. $S \subset V(G)$ is a *separator* (of G) if $\langle V(G) \setminus S\rangle$ is not connected. We denote by $\kappa(G)$ the connectivity of G, namely the minimum cardinality of a separator of G (if G is not a complete graph).

Consider now $H \subset J \subset G$, $Y \subset J \subset G$ with $H \cap Y = \emptyset$, $x \in V(H)$, and $y \in V(Y)$. An *xy-path* P in J is a path in J connecting x and y, while the *length of the path* P is equal to $|E(P)|$. Furthermore, let the *distance* between H and Y in J be defined as

$$d^J(H, Y) = \min_{x\in V(H), y\in V(Y)} \{|E(P)| : P \ xy\text{-path in } J\}.$$

When there exists no path between H and Y in J, we put $d^J(H, Y) = \infty$. When $J = G$ we write $d^G(H, Y) = d(H, Y) = d_H(Y)$ for the distance between H and Y, and call d_H the *distance function* (with respect to H) — in this paper we will use both the second ($d(H, Y)$) and third ($d_H(Y)$) notation, depending on which seems more intuitive in a given context.

The *eccentricity* of a vertex x, and the *radius* and *diameter* of G are defined as

$$\varepsilon(x) = \max_{y\in V(G)} d(x, y), \qquad r(G) = \min_{x\in V(G)} \varepsilon(x), \quad \text{and} \quad \operatorname{diam}(G) = \max_{x\in V(G)} \varepsilon(x),$$

respectively. For a survey on diameters of graphs, see [15]. An important article on radii, diameters and minimum degree is [17]. The *centre* of a graph is $Z(G) = \{x : \varepsilon(x) = r(G)\}$. A graph is *self-centred* (also *equi-eccentric*), if $r(G) = \operatorname{diam}(G)$. Denote the set of all self-centred graphs by $\mathcal{C}$. For a survey on the family of graphs $\mathcal{C}$, see [6].

We recall [27] that a finite metric space (X, ρ) is isometric to some graph if and only if (i) for any two points $x, y \in X$ we have $\rho(x, y) \in \mathbb{N}_0$, and (ii) if $\rho(x, y) \geq 2$ then there exists a third point z such that $\rho(x, z) + \rho(z, y) = \rho(x, y)$.

A shortest path σ between x and y, i.e. an xy-path of length $d(x, y)$, will be called an *xy-segment*, and its length denoted by $|\sigma|$. (Note that many graph theorists call the shortest path between two vertices "geodesic", but we opine that, especially when writing a paper concerning the cut locus, a notion from Differential Geometry, it is acceptable to deviate from standard graph-theoretical nomenclature.) In Geometry, this corresponds to the term "segment", or "geodesic segment", or "minimizing geodesic", i.e. a globally distance-minimizing arc. A graph theoretical notion corresponding to a "geodesic" (i.e. locally distance minimizing) can easily be defined, for instance by asking a path to admit no chord between any two of its vertices at distance 2 on the path, but will not be used in this paper.

Consider a subgraph $H \subset G$ and a vertex $y \in V(G)$. An *Hy-segment* is a path between some $x \in V(H)$ and y of length $d_H(y)$. Put

$$N_i(H) = \{y \in V(G) : d_H(y) = i\} \quad \text{and} \quad N_i[H] = \{y \in V(G) : d_H(y) \leq i\}, \quad i \geq 0.$$

$N_1(H) = N(H)$ ($N_1[H] = N[H]$) is the *neighbourhood* (*closed neighbourhood*) of H in G. When two vertices are at distance 1, we say that they are *neighbours*. For a vertex $x \in V(H)$ define its *degree* (with respect to H) as $\deg_H(x) = |\{y \in V(H) : d^H(x, y) = 1\}|$, and put $\deg_G(x) = \deg(x)$. Furthermore, let us write $V_1(H) = \{x \in V(H) : \deg(x) = 1\}$. Define

$$F(H) = \left\langle \left\{ x \in V(G) : d_H(x) = \max_{y\in V(G)} d_H(y) \right\} \right\rangle,$$

which is the graph induced by the vertices realizing global maxima of the distance function d_H, i.e. the graph induced by the *farthest vertices from* H. Two points $x, y \in V(G)$ are *antipodal* (each is the antipode of the other) if $d(x, y) = \operatorname{diam}(G)$. The function F is the *farthest point mapping*. Notice that for a vertex x we have $\varepsilon(x) = d(x, F(x))$. For a local version, put

$$M(H) = \langle\{x \in V(\complement H) : d_H(x) \geq d_H(y) \ \forall y \in N(x)\}\rangle \supset F(H)$$

and, for $i \geq 1$,

$$M_i(H) = \{v \in N_i(H) : E(v, N_{i+1}(H)) = \emptyset\}.$$

[1] Extending the results to multigraphs would result in additional technicalities in some proofs.