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Spanning trees with nonseparating paths ☆

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ABSTRACT

We consider questions related to the existence of spanning trees in connected graphs with the property that, after the removal of any path in the tree, the graph remains connected. We show that, for planar graphs, the existence of trees with this property is closely related to the Hamiltonicity of the graph. For graphs with a 1- or 2-vertex cut, the Hamiltonicity also plays a central role. We also deal with spanning trees satisfying this property restricted to paths arising from fundamental cycles. The cycle space of a graph can be generated by the fundamental cycles of every spanning tree, and Tutte showed that, for a 3-connected graph, it can be generated by nonseparating cycles. We are also interested in the existence of a fundamental basis consisting of nonseparating cycles.

1. Introduction

In this paper every graph is finite, simple, and connected. The *vertex set* of a graph G is denoted by $V(G)$ and its *edge set* by $E(G)$. For every subgraph H of G, $G[H]$ is the subgraph of G induced by $V(H)$, and $G - H$ is the subgraph of G induced by $V(G) \setminus V(H)$. We say G is *k-connected* if $|V(G)| > k$ and $G - X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$. By convention, both the *null graph* (the graph with no vertices, and hence no edges) and the *trivial graph* (every graph with just one vertex) are 0- and 1-connected, but are not k-connected for each $k > 1$.

If G is a graph and H a subgraph of G, we say that H is *nonseparating (separating)* if $G - H$ is connected (respectively, disconnected). A path P that links u with v is called a *uv-path*. Tutte [12] proved that, for every 3-connected graph G and vertices u and v, there exists a nonseparating uv-path. In 1975, Lovász [10] made the following conjecture which is related to this result of Tutte.

Conjecture 1. *For every positive integer k, there exists a positive integer $f(k)$ such that, for every $f(k)$-connected graph G and vertices u and v, there exists a uv-path P such that $G - P$ is k-connected.*

It is easy to see that $f(1) \geq 3$. So, Tutte's result implies that $f(1) = 3$. Chen, Gould, and Yu [3], and independently Kriesell [9], proved that $f(2) = 5$. Recently, Kawarabayashi, Lee, and Yu [7] proved that $f(2) = 4$ except for double wheels. The conjecture is open for $k \geq 3$.

Some related questions have been settled yielding new conjectures. One of them is the following due to Kawarabayashi and Ozeki [8].

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Conjecture 2. *For all positive integers k and ℓ, there exists a positive integer $g(k,\ell)$ such that the following holds. For every $g(k,\ell)$-connected graph G and vertices u and v, there exist internally disjoint uv-paths $P_1,\ldots,P_\ell$ such that $G-\bigcup_{i=1}^{\ell} P_i$ is k-connected.*

For $\ell = 1$, this corresponds to Lovász Conjecture. Kawarabayashi and Ozeki showed that $g(1,\ell) = 2\ell+1$ and $g(2,\ell) \leq 3\ell+2$.

Another related question was raised by Hong and Lai [6], who considered the problem of connecting a subset of vertices by a tree instead of connecting just two vertices as in Tutte's theorem. They made the conjecture below.

Conjecture 3. *For all positive integers k and r, there exists a positive integer $h(k,r)$ such that, for every $h(k,r)$-connected graph G and subset X with r vertices, there exists a tree T connecting X such that $G-T$ is k-connected.*

Hong and Lai proved that $h(1,r) = r+1$ and $h(2,r) \leq 2r+1$.

By Tutte's theorem, in a 3-connected graph, every pair of vertices is connected by a nonseparating path. *Is it possible to get a spanning tree in which all paths are nonseparating?* Inspired by Tutte's result, we say that a spanning tree of a graph is a *Tutte tree* if every path in the tree is nonseparating. In this paper we deal with the following question.

Question 4. Which graphs have a Tutte tree?

We prove the following results.

Theorem 5. *A planar graph has a Tutte tree if and only if it is Hamiltonian or it has a spanning tree whose leaves induce a triangle.*

Theorem 6. *Let G be a graph and T be a spanning tree. If the subgraph induced by the leaves of T is 3-connected, then T is a Tutte tree.*

Let T be a spanning tree of G. For any two vertices u and v, there exists a unique uv-path in T, denoted by uTv. For every $e \in E(G) \setminus E(T)$, there is a unique cycle C_e in $T+e$. These cycles C_e are called *fundamental cycles* (of G) with respect to T.

Tutte [12] proved that the cycle space of a 3-connected graph is generated by its nonseparating induced cycles. Once more, inspired by a result of Tutte, we say that a spanning tree T of a graph is a *fundamental Tutte tree* if each fundamental cycle with respect to T is nonseparating. We adopt the convention that a Hamiltonian cycle is nonseparating.

We present a question concerning fundamental Tutte trees similar to Question 4.

Question 7. Which graphs have a fundamental Tutte tree?

This question corresponds to asking if there is a fundamental basis of the cycle space consisting of nonseparating cycles.

For the case of planar graphs, we use the same terminology as Tutte [13]. Let G be a planar graph and D be a plane drawing of G. Consider a set S of vertices of G. The vertices in S are *face-neighbors* in D if some face of D has all vertices in S in its boundary.

Theorem 8. *Let G be a planar graph and T be a spanning tree. If T is a fundamental Tutte tree, then the leaves of T are face-neighbors in every plane drawing of G.*

It is easy to see that if T is a Tutte tree of G, then T is a fundamental Tutte tree. However, as we shall see, not every graph with a fundamental Tutte tree has a Tutte tree.

This paper is structured as follows. In Section 2, we dive into Tutte trees. The structure of graphs with a 2-vertex cut having a Tutte tree is investigated. We analyze Tutte trees in planar graphs and prove Theorem 5. Some examples of graphs with no Tutte tree are presented. We show that the problem of deciding whether a 3-connected graph has a Tutte tree is NP-complete. We prove Theorem 6 and exhibit examples showing that the sufficient condition of this theorem is not a necessary one. Next, in Section 3, we contemplate the fundamental Tutte trees. We explore the structure of graphs having a fundamental Tutte tree with a 1- or a 2-vertex cut. Some results related to separating cycles in planar graphs are proved in preparation for proving Theorem 8. An example of a graph with a fundamental Tutte tree and no Tutte tree is presented. We conclude this section presenting a graph with no fundamental Tutte tree. Finally, in Section 4, we present some concluding remarks and open questions.

2. Tutte trees

Note that 3-connectedness is a sufficient condition for a graph to have a nonseparating path linking any two vertices. However, it is not a necessary condition. A cycle of length at least three, which is 2-connected but not 3-connected, has a nonseparating path linking any two vertices. The following observation is trivial, but important.

Observation 9. *Every graph with at least three vertices and a Tutte tree is 2-connected.*