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Maximum uniformly resolvable decompositions of K_v and $K_v - I$ into 3-stars and 3-cycles

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ABSTRACT

Let K_v denote the complete graph of order v and $K_v - I$ denote K_v minus a 1-factor. In this article we investigate uniformly resolvable decompositions of K_v and $K_v - I$ into r classes containing only copies of 3-stars and s classes containing only copies of 3-cycles. We completely determine the spectrum in the case where the number of resolution classes of 3-stars is maximum.

1. Introduction

Given a collection of graphs $\mathcal{H}$, an *$\mathcal{H}$-decomposition* of a graph G is a decomposition of the edges of G into isomorphic copies of graphs from $\mathcal{H}$, the copies of $H \in \mathcal{H}$ in the decomposition are called *blocks*. Such a decomposition is called *resolvable* if it is possible to partition the blocks into *classes* $\mathcal{P}_i$ such that every point of G appears exactly once in some block of each $\mathcal{P}_i$.

A resolvable $\mathcal{H}$-decomposition of G is sometimes also referred to as an *$\mathcal{H}$-factorization* of G, a class can be called a *$\mathcal{H}$-factor of G*. The case where $\mathcal{H}$ is a single edge (K_2) is known as a 1-*factorization of* G and it is well known to exist for $G = K_v$ if and only if v is even. A single class of a 1-factorization, a pairing of all points, is also known as a 1-*factor* or a *perfect matching*.

In many cases we wish to impose further constraints on the classes of an $\mathcal{H}$-decomposition. For example, a class is called *uniform* if every block of the class is isomorphic to the same graph from $\mathcal{H}$. Of particular note is the result of Rees [11] which finds necessary and sufficient conditions for the existence of uniform $\{K_2, K_3\}$-decompositions of K_v. Uniformly resolvable decompositions of K_v have also been studied in [3,6,7,9,10,13,14]. Moreover, recently Dinitz, Ling and Danziger [4] have solved the question of the existence of a uniformly resolvable decomposition of K_v into r classes of K_2 and s classes of K_4 *in the case in which the number s of* K_4*-factors is maximum.*

1.1. Definitions and notation

For any four vertices a_1, a_2, a_3, a_4, let the 3-*star*, $K_{1,3}$, be the simple graph with the vertex set $\{a_1, a_2, a_3, a_4\}$ and the edge set $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}\}$. In what follows, we will denote it by $(a_1; a_2, a_3, a_4)$.

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Let $K_{m(n)}$ denote the complete multipartite graph with m parts each of size n, that is, $K_{m(n)}$ has the vertex set $\{\bigcup_{i=1}^{m} X^i\}$ with $|X^i| = n$ for $i = 1, 2, \ldots, m$ and $X^i \cap X^j = \emptyset$ for $i \neq j$, and the edge set $\{\{u, v\} : u \in X^i,\ v \in X^j,\ 1 \leq i < j \leq m\}$.

Let $C_{m(n)}$ denote the graph with the vertex set $\{\bigcup_{i=1}^{m} X^i\}$ with $|X^i| = n$ for $i = 1, 2, \ldots, m$ and $X^i \cap X^j = \emptyset$ for $i \neq j$, and the edge set $\{\{u, v\} : u \in X^i,\ v \in X^j,\ i - j \equiv 1 \pmod{m}$ or $j - i \equiv 1 \pmod{m}\}$. For constructions below we shall also need the particular case $|X^i| = 12$. Then let $X^i = \{x_h^i : h = 0, 1, \ldots, 11\}$ and for each $j \in \{0, 1, 2, 3\}$ let $X_j^i = \{x_{3j}^i, x_{3j+1}^i, x_{3j+2}^i\}$, so that $X^i = \bigcup_{j=0}^{3} X_j^i$. Define, for each $i = 1, 2, \ldots, m$ and $r, s \in \{0, 1, 2, 3\}$, the following sets of 3-stars:

$$\begin{aligned} R_{r,s}^i &= \{X_r^i; X_s^{i+1}, X_{s+1}^{i+1}, X_{s+2}^{i+1}\} \\ &= \{\{x_{3r}^i; x_{3s}^{i+1}, x_{3s+1}^{i+1}, x_{3s+2}^{i+1}\}, \{x_{3r+1}^i; x_{3s+3}^{i+1}, x_{3s+4}^{i+1}, x_{3s+5}^{i+1}\}, \{x_{3r+2}^i; x_{3s+6}^{i+1}, x_{3s+7}^{i+1}, x_{3s+8}^{i+1}\}\} \end{aligned}$$

where superscript addition is meant modulo 12.

A resolvable $\mathcal{H}$-decomposition of $K_{m(n)}$ is known as a *resolvable group divisible design $\mathcal{H}$-RGDD of type n^m*, where the parts of size n are called the groups of the design. When $\mathcal{H} = K_n$ we will call it an n-RGDD. We shall use the terms "point" and "vertex" as synonyms.

1.2. Our results

In this paper we study the existence of a uniformly resolvable decomposition of K_v and of $K_v - I$, having the following type:

r classes containing only copies of 3-stars and s classes containing only copies of 3-cycles.

We will use the notation $(K_{1,3}, K_3)$-URD$(v; r, s)$ for such a uniformly resolvable decomposition of K_v when v is odd, and for that of $K_v - I$ when v is even. We will specify whether the system is a decomposition of K_v or of $K_v - I$ only when it is not clear in the context whether v is odd or even. Further, we will use the notation $(K_{1,3}, K_3)$-URGDD(r, s) of $K_{m(n)}(C_{m(n)})$ to denote a uniformly resolvable decomposition of $K_{m(n)}(C_{m(n)})$ into r classes containing only copies of 3-stars and s classes containing only copies of 3-cycles. As r determines s if v is fixed, we will also use the simplified notation $K_{1,3}$-RGDD(r) for $(K_{1,3}, K_3)$-URGDD(r, s) when v is understood.

Determining the spectrum of triples (v, r, s) which admit a $(K_{1,3}, K_3)$-URD$(v; r, s)$ appears to be a rather hard problem in general. Similar to the work in [4], here we concentrate on the extremal case in which the number of resolution classes of 3-stars is maximum. In particular, we will prove the following result in this paper:

Main Theorem. *For each $v \equiv 0 \pmod{12}$, there exists a $(K_{1,3}, K_3)$-URD $(v; \frac{2(v-6)}{3}, 2)$ of $K_v - I$.*

2. Necessary conditions

In this section we will give necessary conditions for the existence of a uniformly resolvable decomposition of K_v and $K_v - I$ into r classes of 3-stars and s classes of 3-cycles.

Lemma 2.1. *A $(K_{1,3}, K_3)$-URD$(v; r, s)$, with $r > 0$ and $s > 0$, does not exist for any $v \geq 4$ of K_v.*

Proof. Assume that there exists a $(K_{1,3}, K_3)$-URD$(v; r, s)$ D of K_v with $r > 0$ and $s > 0$. By resolvability it follows that $v \equiv 0 \pmod{12}$, say $v = 12u$. Counting the edges of K_v that appear in D we obtain

$$\frac{3rv}{4} + \frac{3sv}{3} = \frac{v(v-1)}{2}$$

and hence

$$3r + 4s = 2(v - 1). \tag{1}$$

The equality (1) implies that $r \equiv 2 \pmod{4}$ and $s \equiv 1 \pmod{3}$. Let $r = 2 + 4t$ and $s = 1 + 3h$ with $t, h \geq 0$. Denote by B the set of the r parallel classes of 3-stars and by R the set of the s parallel classes of 3-cycles. Since the classes of R are regular of degree 2, we have that every vertex x of K_v is incident with $2s$ edges in R and $(12u - 1) - (2 + 6h) = 12u - 6h - 3$ edges in B. Assume that the vertex x appears in a classes with degree 3 and in b classes with degree 1 in B. Since

$$a + b = 2 + 4t \quad \text{and} \quad 3a + b = 12u - 6h - 3,$$

it follows that

$$2a = 12u - 6h - 3 - 2 - 4t = 2(6u - 3h - 1 - 2t) - 3,$$

which is a contradiction, since $2a$ cannot be odd. □

Given $v \equiv 0 \pmod{12}$, define

$$J(v) = \left\{ \left(4x, \frac{v-2}{2} - 3x\right) : x = 1, \ldots, \frac{v-6}{6} \right\}.$$

Lemma 2.2. *If there exists a $(K_{1,3}, K_3)$-URD$(v; r, s)$ of $K_v - I$ with $r > 0$ and $s > 0$ then $v \equiv 0 \pmod{12}$ and $(r, s) \in J(v)$.*