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Forbidden subgraphs for longest cycles to contain vertices with large degrees[☆]

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ABSTRACT

Let G be a graph. For a given graph H, we say that G is H-*free* if G contains no copies of H as an induced subgraph. Suppose that G is 2-connected, has n vertices, and α is a real number with $0 \leq \alpha \leq 1$. In this paper, we characterize the connected graphs R such that G being R-free implies that every longest cycle of G passes through all vertices with degree at least $\alpha n + O(1)$ in G.

1. Introduction

We use Bondy and Murty [4] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph. For a vertex $v \in V(G)$ and a subgraph H of G, we use $N_H(v)$ to denote the set, and $d_H(v)$ the number, of neighbors of v in H. We call $d_H(v)$ the *degree* of v in H. For two subgraphs H and L, we set $N_L(H) = \bigcup_{v\in V(H)} N_L(v)$. When no confusion occurs, we will denote $N_G(v)$ and $d_G(v)$ by $N(v)$ and $d(v)$, respectively.

A graph is called *hamiltonian* if it contains a cycle passing through all its vertices (a Hamilton cycle). We first give a result on hamiltonian graphs.

Theorem 1 (*Dirac [7]*). *Let G be a graph on $n \geq 3$ vertices. If every vertex of G has degree at least $n/2$, then G is hamiltonian.*

Let G be a graph on n vertices. A vertex with degree at least $n/2$ is called a *heavy vertex*, and a cycle containing all heavy vertices is called a *heavy cycle* of G. Dirac's theorem means that if every vertex of G is heavy, then G contains a Hamilton cycle (which is a heavy cycle). This result was extended to the following

Theorem 2 (*Bollobás and Brightwell [3], Shi [14]*). *Every 2-connected graph has a heavy cycle.*

In general graphs, a longest cycle is not necessarily a heavy cycle. So a natural problem is: In which graphs, all the longest cycles are heavy?

In this paper, we consider a more general problem: Under what conditions, all longest cycles in a graph contain every vertex of degree with a lower bound?

To avoid discussions of trivial cases, we put our consideration on 2-connected graphs. The graph on n vertices in Fig. 1 shows that a vertex with degree $n - 5$ is not necessarily contained in a longest cycle. On the other hand, for any graph on $n \geq 8$ vertices, we can prove that every vertex with degree at least $n - 4$ is contained in every longest cycle.

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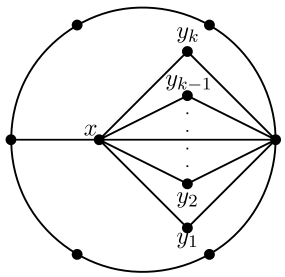

Fig. 1. A graph with a longest cycle excluding a vertex with degree $n-5$ ($k=n-7$).

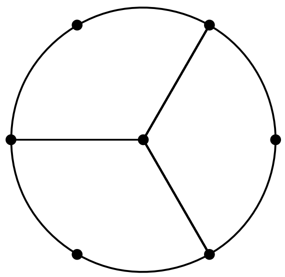

Fig. 2. A graph on 7 vertices with a longest cycle excluding a vertex with degree 3.

Theorem 3. *Let G be a 2-connected graph on $n \geq 8$ vertices, and C be a longest cycle of G. Then C contains every vertex with degree at least $n-4$ in G.*

Throughout this paper, for a cycle C of G with a given orientation and a vertex v in C, we use v^+ to denote the immediate successor, and v^- the immediate predecessor, of v on C.

Proof. Suppose not. Let x be a vertex in $V(G) \setminus V(C)$ such that $d(x) \geq n-4$. We will get a contradiction. Note that there are at most three vertices in $V(G) \setminus \{x\}$ that are not adjacent with x. If x has more than 3 neighbors on C, then there are two neighbors y, y' of x such that $yy' \in E(C)$. Let C' be the cycle obtained from C by replacing the edge yy' with the path yxy'. Then C' is a cycle longer than C, a contradiction. This implies that x has at most three neighbors on C.

Let H be the component of $G-C$ containing x. Since $n \geq 8$ and $d(x) \geq n-4$, x has a neighbor x' in H. Since G is 2-connected, there are two disjoint paths P and P' from x and x', respectively, to C. Let y and z be the end vertices of P and P', respectively, on C.

We give an orientation to C. If there is a path P'' from x to z^{--} with all internal vertices in $H-P'$, then let C' be the cycle obtained from C by replacing the subpath $z^{--}z^-z$ with the path $P''xx'P'$. Then C' is a cycle longer than C, a contradiction. This implies that $y \neq z^{--}$ and $xz^{--} \notin E(G)$. Similarly, we can prove that $y \neq z^-, z^+, z^{++}$ and $xz^-, xz^+, xz^{++} \notin E(G)$. This implies that z^{--}, z^-, z^+, z^{++} are four distinct vertices of C nonadjacent to x, a contradiction. □

The condition $n \geq 8$ in Theorem 3 is necessary. A counterexample on 7 vertices is shown in Fig. 2.

Let G be a graph and G' be a subgraph of G. If G' contains all edges $xy \in E(G)$ with $x, y \in V(G')$, then G' is called an *induced subgraph* of G (or a subgraph *induced by* $V(G')$). For a given graph H, we say that G is *H-free* if G does not contain an induced subgraph isomorphic to H. If G is H-free, then we call H a *forbidden subgraph* of G. Note that if H_1 is an induced subgraph of H_2, then an H_1-free graph is also H_2-free.

Forbidden singletons and forbidden pairs of connected graphs that imply that a 2-connected graph is hamiltonian have been characterized. Also, similar characterizations have been given for other hamiltonian properties such as traceability, and pancyclicity, see [1,10].

Let α be a real number with $0 \leq \alpha \leq 1$. In this paper, we will consider that which forbidden subgraphs R can guarantee that a 2-connected graph G of order n being R-free implies that any longest cycle of G passes through all the vertices with degree at least $\alpha n + O(1)$ (that is to say, there exists a constant A such that any longest cycle of G contains all the vertices with degree at least $\alpha n + A$). First note that if $\alpha = 1$, then by Theorem 3, every graph meets our result. Now we assume that $\alpha < 1$. Note that a K_2-free graph is an empty graph (contains no edges). To avoid the discussion of this case, we assume all the forbidden subgraphs considered have at least three vertices.

The graph $K_{1,r}$ ($r \geq 2$) is called a *star*. Its only vertex with degree r is called the *center* and the other vertices are the *end vertices* of the star. Thus a star with two end vertices is the path on three vertices (denoted by P_3), and we call the star with three end vertices ($K_{1,3}$) a *claw*.

Theorem 4. *Let α be a real number with $0 \leq \alpha < 1$, G be a 2-connected graph on n vertices, and R be a connected graph on at least three vertices. Then G being R-free implies every longest cycle of G contains all the vertices with degree at least $\alpha n + O(1)$ in G, if and only if R is a star $K_{1,r}$ with $r = 2, \ldots, \lfloor 2/(1-\alpha) \rfloor$.*