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# $d$-strong total colorings of graphs

Arnfried Kemnitz *, Massimiliano Marangio

*Computational Mathematics, Technische Universität Braunschweig, Rebenring 31, 38106 Braunschweig, Germany*

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ABSTRACT

For a proper total coloring of a graph $G = (V, E)$ the palette $C(v)$ of a vertex $v \in V$ is the set of the colors of the incident edges and the color of the vertex itself. If $C(u) \neq C(v)$ then the two vertices $u$ and $v$ of $G$ are distinguished by the total coloring. A $d$-strong total coloring of $G$ is a proper total coloring that distinguishes all pairs of vertices $u$ and $v$ with distance $1 \leq d(u, v) \leq d$. The $d$-strong total chromatic number $\chi''_d(G)$ of $G$ is the minimum number of colors of a $d$-strong total coloring of $G$. Such total colorings generalize strong total colorings and adjacent strong total colorings as well.

In this paper we present general lower bounds, determine $\chi''_d(G)$ completely for paths and give exact values and bounds for cycles and for circulant graphs. Moreover, we disprove a conjecture on a general upper bound for $\chi''_d(G)$ (Zhang et al., 2006).


## 1. Introduction

If $c : V \cup E \to \{1, 2, \ldots, k\}$ is a proper total coloring of a graph $G = (V, E)$ then the *palette* or *color set* $C(v)$ of a vertex $v \in V$ is the set of colors of the incident edges and the color of $v$: $C(v) = \{c(e) : e = vw \in E\} \cup \{c(v)\}$. A total coloring $c$ *distinguishes* vertices $u$ and $v$ if $C(u) \neq C(v)$.

The *strong total chromatic number* $\chi''_s(G)$ or *vertex distinguishing total chromatic number* $\chi_{vt}(G)$ is the minimum number of colors of a proper total coloring of a graph $G$ that distinguishes every pair of distinct vertices. Such total colorings are called *strong total colorings* or *vertex distinguishing total colorings* (*VDTC*) and were introduced in [18] (for results see also [1,2]).

On the other hand, the *adjacent strong total chromatic number* or *adjacent vertex distinguishing total chromatic number* $\chi''_{as}(G)$, $\chi''_a(G)$, or $\chi_{at}(G)$ is defined as the minimum number of colors of a proper total coloring of a graph $G$ that distinguishes every pair of adjacent vertices. This parameter was first studied in [3,16]. Further results can be found in, e.g., [4–6,8,9,11].

In this paper we consider *$d$-strong total colorings* of a graph $G$. Such a coloring is a proper total coloring of $G$ that distinguishes all pairs of vertices $u$ and $v$ with distance $1 \leq d(u, v) \leq d$. The minimum number of colors of a $d$-strong total coloring is called *$d$-strong total chromatic number* $\chi''_d(G)$ of $G$. These total colorings and chromatic invariants were introduced by Zhang et al. in [17] as *$D(d)$-vertex distinguishing total colorings* ($D(d)$-*VDTC*) and *$D(d)$-vertex distinguishing total chromatic numbers* $\chi_{dvt}(G)$, respectively. Results for small $d$ are also contained in [15,19].

As an example, a 1-strong total coloring of the complete bipartite graph $K_{2,3}$ with 4 colors and a 2-strong (also strong) total coloring with 5 colors are shown in Fig. 1. Adjacent vertices of $K_{2,3}$ have palettes of different cardinality which implies that every total coloring is also a 1-strong total coloring. Since $\Delta(K_{2,3})+1 = 4$ is a trivial lower bound for $\chi''_1(K_{2,3})$ it follows that $\chi''_1(K_{2,3}) = 4$. On the other hand, $\chi''_2(K_{2,3}) \geq 5$ since at least 5 colors are needed to distinguish the two vertices of degree 3 at distance 2 from each other. Therefore $\chi''_2(K_{2,3}) = 5$.

* Corresponding author.
*E-mail addresses:* a.kemnitz@tu-bs.de (A. Kemnitz), m.marangio@tu-bs.de (M. Marangio).

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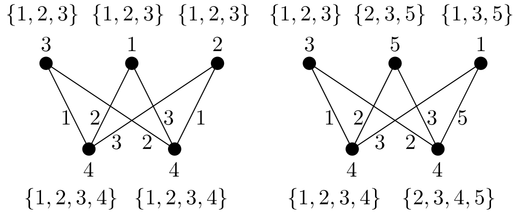


**Fig. 1.** $\chi_1''(K_{2,3}) = 4$, $\chi_2''(K_{2,3}) = 5$.

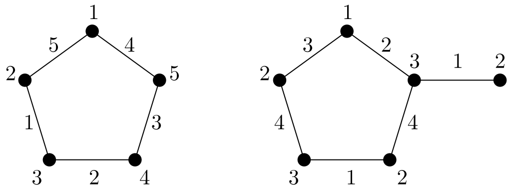

**Fig. 2.** $\chi_2''(C_5) = 5$, $\chi_2''(H) = 4$ where $C_5 \subseteq H$.

Note that $\chi_{as}''(G) = \chi_1''(G)$ and that for connected graphs $\chi_s''(G) = \chi_d''(G)$ if $d \geq \mathrm{diam}(G)$ where diam $(G)$ is the *diameter* of the graph $G$. Therefore, $d$-strong total colorings are generalizations of strong total colorings and of adjacent vertex distinguishing total colorings as well.

The following properties of $\chi_d''(G)$ are obvious.

**Lemma 1** (*Monotonicity*). *If $d \leq t$ then $\chi_d''(G) \leq \chi_t''(G)$.*

**Proof.** A $t$-strong total coloring of $G$ with $t \geq d$ clearly distinguishes all pairs of vertices of distance at most $d$. □

Obviously, it also holds $\chi_d''(G) \leq \chi_s''(G)$ for any $d \in \mathbb{N}$.

**Lemma 2** (*Additivity*). *If $G = H_1 \cup H_2$ then $\chi_d''(G) = \max\{\chi_d''(H_1), \chi_d''(H_2)\}$.*

**Proof.** The components of $G$ can be colored independently since vertices in different components have not to be distinguished. □

If $G$ is a subgraph of $H$, $G \subseteq H$, then note that this does not imply $\chi_d''(G) \leq \chi_d''(H)$ in general (see Fig. 2 for a counterexample), that is, the property $\chi_d''(G) \leq k$ is not a hereditary property (also not an induced hereditary property by the example).

## 2. General bounds

Let $G$ be a graph of order $n$ and let $d_i$ be the number of vertices of $G$ of degree $i$. Then

$$\mu_s''(G) = \max\left\{\min\left\{j : \binom{j}{i+1} \geq d_i\right\} : \delta(G) \leq i \leq \Delta(G)\right\}$$

is a trivial lower bound for the strong total chromatic number, $\chi_s''(G) \geq \mu_s''(G)$. It is conjectured that also $\chi_s''(G) \leq \mu_s''(G)+1$ holds, i.e., that the strong total chromatic number of a graph $G$ attains one of two values. This conjecture is true, e.g., for paths $P_n$, cycles $C_n$, complete graphs $K_n$, and complete bipartite graphs $K_{n,m}$ (see [16,18]). A general upper bound for the strong total chromatic number is $\chi_s''(G) \leq |V(G)| + 2$ [18].

Analogously, let in the following $n_i$ denote the maximum number of vertices of degree $i$ that are of pairwise distance at most $d$. Then

$$\mu_d''(G) = \max\left\{\min\left\{j : \binom{j}{i+1} \geq n_i\right\} : \delta(G) \leq i \leq \Delta(G)\right\}$$

is a trivial lower bound for the $d$-strong total chromatic number,

$$\chi_d''(G) \geq \mu_d''(G). \tag{1}$$

Note that $n_i = d_i$ and therefore $\mu_d''(G) = \mu_s''(G)$ for connected graphs $G$ and $d \geq \mathrm{diam}(G)$.

Zhang et al. [17] conjectured that also the $d$-strong total chromatic number attains one of two possible values.

**Conjecture 3** ([17]). *$\chi_d''(G) \leq \mu_d''(G) + 1$ for all connected graphs $G$ with $|V(G)| \geq 2$.*

Conjecture 3 implies $\chi_1''(G) \leq \Delta(G) + 3$, a well-known conjecture for the adjacent strong total chromatic number (see [16]), since there are at most $i+1$ pairwise adjacent vertices of degree $i$ in $G$ which implies $\mu_1''(G) \leq \Delta(G)+2$ and therefore,