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Locating–paired-dominating sets in square grids

Ľudovít Niepel

Computer Science Department, Kuwait University, P.O. Box 5969, Safat, 13060, Kuwait

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ABSTRACT

A set S of vertices of a graph G is paired-dominating if S induces a matching in G and S dominates all vertices of G. A set $S \subset V(G)$ is locating if for any two distinct vertices $u, v \in V(G) \setminus S$, $N(u) \cap S \neq N(v) \cap S$, where $N(u)$ and $N(v)$ are open neighborhoods of vertices u and v. We give a complete characterization of locating–paired-dominating sets with minimal density in the infinite square grid $\mathbb{Z}^2$.

1. Introduction

The concept of locating–paired-dominating sets was introduced in [16] as an extension of paired-dominating sets [11,12]. The location of monitoring devices in a system when every monitor is paired with a backup monitor serves as the motivation for this concept.

A set S of vertices of a graph $G = (V, E)$ is a *dominating set* of G if every vertex in $V \setminus S$ is adjacent to a vertex of S and S is a *total dominating set* if every vertex in V is adjacent to a vertex in S. The domination and its variants are subject of an extensive study. The survey of related topics can be found in [9,8]. For any vertex $v \in V$ the *open neighborhood* is defined as $N_G(v) = \{x \in V : vx \in E\}$, the *closed neighborhood* is defined as $N_G[v] = N_G(v) \cup \{v\}$. A dominating set is called *locating–dominating* if for any pair of distinct vertices $u, v \in V \setminus S$ holds $N_G(u) \cap S \neq N_G(v) \cap S$. A *matching* $M \subset E$ is a set of edges in graph $G(V, E)$ which do not have any vertices in common. A *paired-dominating set*, abbreviated PDS, of a graph G is a set S of vertices of G such that subgraph $G[S]$ induced by S is a matching and S is dominating in G. The set S is said to be a *locating–paired-dominating set*, abbreviated LPDS, if it is PDS, and S is a locating–dominating set. The minimum cardinality of LPDS in a graph G, *locating–paired-domination number*, is denoted as $\gamma_{pr}^{L}(G)$. Locating–dominating sets are also referred as *locating–dominating codes*. The problem of finding an optimal locating–dominating code in arbitrary network is NP-complete [2]. Optimal locating codes for special classes of graphs were studied in [1,3,5–7,18]. Locating–dominating sets in infinite grids and their density were studied in [14,15,17,19]. Locating–total dominating sets in trees are studied in [4,10,13]. Only few results are known for locating–paired-dominating sets in graphs. The estimation of the minimal cardinality of the dominating set and exact results for trees can be found in [16].

In this paper we give a complete characterization of optimal locating–paired-dominating sets in infinite square grids.

2. Preliminaries

For notation and graph theory terminology, we in general follow [9]. All graphs considered in this paper are subgraphs of infinite grid $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$. Vertices (i, j) and (m, n) are adjacent in $\mathbb{Z}^2$ if $|i - m| + |j - n| = 1$. Degree of each vertex in $\mathbb{Z}^2$

E-mail addresses: niepel@cs.ku.edu.kw, niepel@gmail.com.

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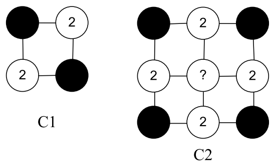

Fig. 1. Forbidden configurations.

is equal to 4, so rectangular grid $\mathbb{Z}^2(V, E)$ is an infinite locally finite 4-regular graph. The density $D(S)$ of a dominating set S in graph $G(V, E)$ is defined as

$$D(S) = \frac{|S|}{|V|}.$$

It is possible to generalize the notion of density of a set to infinite local finite graphs. The *k-neighborhood* of u in G is defined as $N_G^k[u] = \{x \in V : d(u, x) \leqslant k\}$, the set of vertices at distance at most k from u. The *density* of $S \subseteq V$ in V is defined as

$$D(S) = \limsup_{k\to\infty} \frac{\left|S \cap N_G^k[u]\right|}{\left|N_G^k[u]\right|}.$$

As an analogue of locating–paired-dominating number in infinite graphs we can consider the smallest density $\sigma(G)$ of a locating–paired-dominating set S in G. To estimate the value of density of LPDS we can use the notion of the *share* of a vertex.

The share of $v \in S$ in a dominating set S is defined in the following way [19]. For each vertex $u \in V$ we denote $n(u) = |N[u] \cap S|$. The share $sh(v)$ is given by

$$sh(v) = \sum_{u\in N[v]} \frac{1}{n(u)}.$$

The share of a vertex is a measure of contribution of this vertex to domination. The meaning of share of vertices in finite graphs explains the following equation.

$$|V| = \sum_{v\in S} sh(v).$$

From this formula there follows a relationship between values of shares of vertices and density of a dominating set in graph G

$$D(S) = \frac{|S|}{\sum\limits_{v\in S} sh(v)}.$$

In other words the density of a dominating set is the reverse of the average value of shares of its vertices.

Lemma 1. *Let S be a LPDS in a graph G with maximum degree Δ, then $D(S) \geqslant \frac{2}{\Delta+2}$.*

Proof. To prove the statement it is enough to show that each vertex in S has share at most $(\Delta + 2)/2$. As the set S induces a matching, each vertex $v \in S$ is adjacent with at least one vertex in S, and contribution of this vertex to $sh(v)$ is at least $1/2$. It follows from the locating property that at most one vertex in $V \setminus S$ can be uniquely dominated by v and contribution of this vertex to $sh(v)$ is 1. All other vertices in $N_G[v]$ are dominated by at least 2 vertices in S and their contribution is at most $1/2$. So $sh(v) \leqslant 1/2 + 1 + (\Delta - 1)/2 = (\Delta + 2)/2$. ■

Our goal is to classify all LPDS with the minimal density in infinite grid $\mathbb{Z}^2$. Each vertex in $\mathbb{Z}^2$ is of degree 4, so the share of any vertex in any LPDS is at most 3, and the density is at least $1/3$. In an optimal LPDS all vertices have share equal to 3, so each vertex in $V \setminus S$ has at most 2 neighbors in S, each vertex $v \in S$ is adjacent with exactly one vertex in S, and there exists exactly one vertex in $V \setminus S$ exclusively dominated by v. An example of an optimal LPDS in a square grid with density $1/3$ is in Fig. 3. In the rest of the paper we shall investigate optimal LPDS in $\mathbb{Z}^2$ only.

3. Optimal LPDS in $\mathbb{Z}^2$

Let S be an optimal LPDS in $\mathbb{Z}^2$, it is possible to assign to all vertices in $V \setminus S$ the number of their neighbors in S. Each vertex has assigned value 1 or 2. By a *configuration* we understand a subgraph of $\mathbb{Z}^2$ with LPDS S together with assignments of vertices implied by the dominating set S. In Fig. 1 are configurations C_1 and C_2, where black vertices are vertices in a dominating set, and assignments of other vertices are given by the domination.

Observation 2. *Let S be an optimal LPDS in $\mathbb{Z}^2$ then configurations C_1 and C_2 are not possible.*

Proof. Two vertices in configuration C_1 have the same set of dominating vertices, so locating property is violated. The middle vertex in configuration C_2 is not dominated, so its assignment cannot be 1 or 2. In both cases S cannot be an optimal LPDS. ■