Discrete Mathematics 338 (2015) 1706–1713

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Rainbow connection and forbidden subgraphs

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ARTICLE INFO

Article history:
Received 27 August 2013
Accepted 10 August 2014
Available online 10 September 2014

Keywords:
Rainbow connection
Edge coloring
Forbidden subgraph

ABSTRACT

A connected edge-colored graph G is rainbow-connected if any two distinct vertices of G are connected by a path whose edges have pairwise distinct colors; the rainbow connection number $\mathrm{rc}(G)$ of G is the minimum number of colors such that G is rainbow-connected. We consider families $\mathcal{F}$ of connected graphs for which there is a constant $k_{\mathcal{F}}$ such that, for every connected $\mathcal{F}$-free graph G, $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$, where $\mathrm{diam}(G)$ is the diameter of G. In this paper, we give a complete answer for $|\mathcal{F}| \in \{1, 2\}$.

1. Introduction

We use [2] for terminology and notation not defined here and consider finite and simple graphs only. To avoid trivial cases, all graphs considered here will be connected with at least one edge.

An edge-colored connected graph G is called *rainbow-connected* if each pair of distinct vertices of G is connected by a rainbow path, that is, by a path whose edges have pairwise distinct colors. Note that the edge coloring need not be proper. The *rainbow connection number* of G, denoted by $\mathrm{rc}(G)$, is the minimum number of colors such that G is rainbow-connected.

The concept of rainbow connection in graphs was introduced by Chartrand et al. in [7]. An easy observation is that if G has n vertices then $\mathrm{rc}(G) \leq n - 1$, since one may color the edges of a given spanning tree of G with different colors and color the remaining edges with one of the already used colors. Chartrand et al. determined the precise value of the rainbow connection number for several graph classes including complete multipartite graphs [7]. The rainbow connection number has been studied for further graph classes in [4,9,11,15] and for graphs with fixed minimum degree in [4,12,17]. See [16] for a survey.

There are various applications for such edge colorings of graphs. One interesting example is the secure transfer of classified information between agencies (see, e. g., [10]).

The computational complexity of rainbow connectivity has been studied in [5,13]. It is proved that the computation of $\mathrm{rc}(G)$ is NP-hard [5,13]. In fact, it is already NP-complete to decide whether $\mathrm{rc}(G) = 2$. It is also NP-complete to decide whether a given edge-colored graph (with an unbounded number of colors) is rainbow-connected [5]. More generally, it has been shown in [13] that for any fixed $k \geq 2$ it is NP-complete to decide whether $\mathrm{rc}(G) = k$.

For the rainbow connection numbers of graphs the following results are known (and obvious).

Proposition A. *Let G be a connected graph of order n. Then*

(i) $1 \leq \mathrm{rc}(G) \leq n - 1$,
(ii) $\mathrm{rc}(G) \geq \mathrm{diam}(G)$,

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http://dx.doi.org/10.1016/j.disc.2014.08.008
0012-365X/

(iii) $\mathrm{rc}(G) = 1$ *if and only if* G *is complete,*
(iv) $\mathrm{rc}(G) = n - 1$ *if and only if* G *is a tree,*
(v) *if* G *is a cycle of length* $n \geq 4$*, then* $\mathrm{rc}(G) = \lceil \frac{n}{2} \rceil$.

Note that the difference $\mathrm{rc}(G) - \mathrm{diam}(G)$ can be arbitrarily large. For $G = K_{1,n-1}$ we have $\mathrm{rc}(K_{1,n-1}) - \mathrm{diam}(K_{1,n-1}) = (n-1) - 2 = n - 3$. Especially, each bridge requires a single color.

Let $\mathcal{F}$ be a family of connected graphs. We say that a graph G is $\mathcal{F}$*-free* if G does not contain an induced subgraph isomorphic to a graph from $\mathcal{F}$. Specifically, for $\mathcal{F} = \{X\}$ we say that G is X*-free*, and for $\mathcal{F} = \{X, Y\}$ we say that G is (X, Y)*-free*. The members of $\mathcal{F}$ will be referred to in this context as *forbidden induced subgraphs*.

Graphs characterized in terms of forbidden induced subgraphs are known to have many interesting properties. Although, in general, there is no upper bound on $\mathrm{rc}(G)$ in terms of $\mathrm{diam}(G)$, and, in bridgeless graphs, by virtue of Theorem F, $\mathrm{rc}(G)$ can be quadratic in terms of $\mathrm{diam}(G)$, it turns out that forbidden subgraph conditions can remarkably lower the upper bound on $\mathrm{rc}(G)$.

Namely, we will consider the following question.

For which families $\mathcal{F}$ *of connected graphs, there is a constant* $k_{\mathcal{F}}$ *such that a connected graph G being* $\mathcal{F}$*-free implies* $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$?

We give a complete answer for $|\mathcal{F}| = 1$ in Section 3, and for $|\mathcal{F}| = 2$ in Section 4.

2. Preliminary results

In this section we summarize some further notations and facts that will be needed for the proofs of our results.

An edge in a graph G is called a *bridge*, if its removal disconnects the graph. A graph with no bridges is called a *bridgeless graph*. An edge is called *pendant edge*, if one of its end vertices has degree one. For two vertices $x, y \in V(G)$, we denote by $\mathrm{dist}(x, y)$ the distance between x and y in G. The diameter and the radius of a graph G will be denoted by $\mathrm{diam}(G)$ and $\mathrm{rad}(G)$, respectively. For $M \subset V(G)$, we use $G[M]$ to denote the induced subgraph of G on M.

For $x \in V(G)$, we use $N_G(x)$ to denote the *neighborhood of* x *in* G and $N_G[x]$ to denote the *closed neighborhood of* x *in* G (i.e., $N_G(x) = \{y \in V(G)|\ xy \in E(G)\}$ and $N_G[x] = N_G(x) \cup \{x\}$). More generally, for sets $A, B \subset V(G)$, we denote $N_G(A) = \cup_{x \in A} N_G(x)$ and $N_B(A) = N_G(A) \cap B$, and for a subgraph $P \subset G$ we write $N_P(A)$ for $N_{V(P)}(A)$ and $N_G(P)$ for $N_G(V(P))$.

A dominating set D in a graph G is called a *two-way dominating set* if D includes all vertices of G of degree 1. In addition, if $G[D]$ is connected, we call D a *connected two-way dominating set*. Note that if $\delta(G) \geq 2$, then every (connected) dominating set in G is a (connected) two-way dominating set.

Theorem B (*[6]*)**.** *If* D *is a connected two-way dominating set in a graph G, then* $\mathrm{rc}(G) \leq \mathrm{rc}(G[D]) + 3$.

The following simple fact is implicit in the proof of Theorem B in [6]. However, since it is not stated explicitly, and since it will be used several times, we state it here, including its (easy) proof.

Proposition C (*[6]*)**.** *Let G be a graph and let* $F \subset G$ *be a connected subgraph of G such that every vertex in* $V(G) \setminus V(F)$ *has at least* 2 *neighbors in F. Then* $\mathrm{rc}(G) \leq \mathrm{rc}(F) + 2$.

Proof. Color the edges of G as follows:

- color the edges of F with colors $1, \ldots, k$, where $k = \mathrm{rc}(F)$,
- for each $x \in V(G) \setminus V(F)$, choose two edges from x to F and color them with colors $k + 1$ and $k + 2$,
- color the remaining edges arbitrarily (e.g., all of them with color $k + 2$).

Then G is rainbow-connected. ∎

For the proofs of Theorems 4 and 6, we will also need the following two facts by Li et al. [14].

Theorem D (*[14]*)**.** *If G is a connected bridgeless graph of diameter* 2*, then* $\mathrm{rc}(G) \leq 5$.

Theorem E (*[14]*)**.** *If G is a connected graph of diameter* 2 *with* $k \geq 1$ *bridges, then* $\mathrm{rc}(G) \leq k + 2$.

For connected bridgeless graphs, the following upper bound on $\mathrm{rc}(G)$ was proved by Basarajavu et al. [1].

Theorem F (*[1]*)**.** *For every connected bridgeless graph G with radius r,*

$$\mathrm{rc}(G) \leq r(r + 2).$$

Moreover, for every integer $r \geq 1$*, there exists a bridgeless graph G with radius r and* $\mathrm{rc}(G) = r(r + 2)$.

3. One forbidden subgraph

In this section, we characterize all connected graphs X such that every connected X-free graph G satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_X$, where k_X is a constant.

Theorem 1. *Let X be a connected graph. Then there is a constant* k_X *such that every connected X-free graph G satisfies* $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_X$*, if and only if* $X = P_3$.