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2-distance colorings of some direct products of paths and cycles

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ABSTRACT

The square of a graph is obtained by adding edges between vertices of distance two in the original graph. The 2-distance coloring problem of a graph is the vertex coloring problem of its square graph. Accordingly the chromatic number of 2-distance coloring is called the 2-distance chromatic number. The 2-distance coloring problem is equivalent to a kind of the distance two labeling problem, the $L(1, 1)$-labeling problem which is motivated by the channel assignment problem. In this paper we find the 2-distance chromatic number of the direct product of two cycles whose numbers of vertices are large enough. Moreover we find that also for the direct product of a path and a cycle.

1. Introduction

A 2-*distance coloring* of a graph $G = (V, E)$ is an integer valued function f on V such that $f(u) \neq f(v)$ whenever u and v are of distance one or two. The 2-distance coloring problem is not only a generalization of the classical graph coloring problem [9] but also a special kind of the graph labeling problem.

The graph labeling problem is motivated by the channel assignment problem, which addresses an assignment of a channel or a frequency to each transmitter in a wireless communication network. The channels assigned to transmitters must satisfy certain distance constraints to avoid interference between nearby transmitters. Because of tremendous increase of the demand calls in the wireless communication networks, we have to find a channel assignment with minimum span. Hale [7] proposed a mathematical modeling for the distance constrained channel assignment problem. He regarded a wireless communication network with the transmitters as a graph with vertices. Griggs and Yeh [6] introduced an $L(j, k)$-*labeling for a graph* $G = (V, E)$, which is a function $f : V \to \mathbb{N} \cup \{0\}$ such that for all pairs of vertices u, v of G, $|f(u) - f(v)| \geq j$ if $\mathrm{dist}(u, v) = 1$ and $|f(u) - f(v)| \geq k$ if $\mathrm{dist}(u, v) = 2$. The *span* of f is the maximum difference $|f(u) - f(v)|$ over all the pairs of vertices u, v of G and is denoted by span (f). The minimum span over all $L(j, k)$-labelings of G is called the $L(j, k)$-*number* and is denoted by $\lambda_{j,k}(G)$. For the $L(j, k)$-labeling problem, the most studied case is where $j = 2$ and $k = 1$ [3,6]. For surveys of $L(j, k)$-labelings, see [1,2,5,28].

The $L(1, 1)$-labeling problem turns out to be the 2-distance coloring problem. The *square* G^2 of a graph G is obtained from G by adding edges between vertices of distance two and the 2-distance coloring problem of G is the vertex coloring problem of G^2. The chromatic number of the 2-distance coloring problem is called the 2-*distance chromatic number* and is denoted

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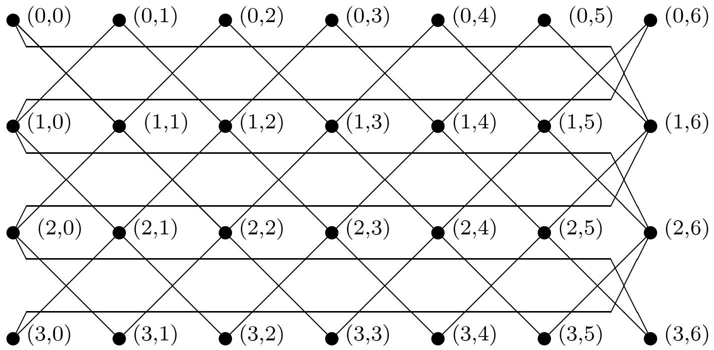

Fig. 1. The direct product $P_4 \times C_7$.

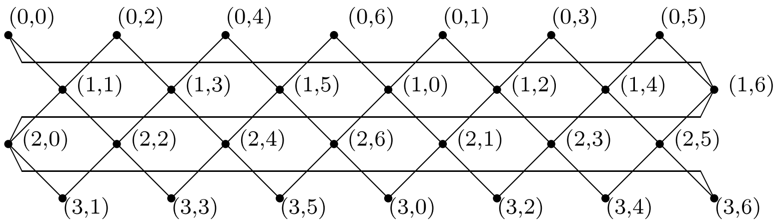

Fig. 2. $G_{4,14} \approx P_4 \times C_7$.

by $\chi_2(G)$. Study of $\chi_2(G)$ for a graph G was initiated by Kramer and Kramer [14,13] in more general context. See [15,18,19,26,30] and a survey paper [16] for this concept. Note that $\chi_2(G) = \lambda_{1,1}(G) + 1$.

The *Cartesian product* $G \square H$ of two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is the graph such that its vertex set is $V_G \times V_H$ and edge set is composed of $\{(u_1, u_2), (v_1, v_2)\}$ where $(\{u_1, v_1\} \in E_G$ and $u_2 = v_2)$ or $(u_1 = v_1$ and $\{u_2, v_2\} \in E_H)$. The *direct product* $G \times H$ of two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is the graph such that its vertex set is $V_G \times V_H$ and edge set is composed of $\{(u_1, u_2), (v_1, v_2)\}$ where $\{u_1, v_1\} \in E_G$ and $\{u_2, v_2\} \in E_H$. The $L(2, 1)$-labeling problem for the Cartesian product [4,11,20,27,29] and the direct product of some families of graphs are studied by [8,11,10,12,17,21,23].

The $L(1, 1)$-number of the Cartesian product of graphs, especially two cycles, are obtained in [22,24]. Also the $L(j, k)$-number of the direct product of a path and a cycle for $2j \le k$ is obtained in [23]. In this paper, we find $\chi_2(C_m \times C_n)$, the 2-distance chromatic number of $C_m \times C_n$ for sufficiently large m and n. In fact, we find 2-distance colorings of some $C_m \times C_n$ and then combine them to find a 2-distance coloring of $C_m \times C_n$ for sufficiently large m and n. Moreover we find $\chi_2(P_m \times C_n)$, the 2-distance chromatic number of $P_m \times C_n$ by a similar method.

2. The 2-distance coloring of the direct product of P_m and C_n

Shiu and Wu [23] noted that if $n \ge 3$, then

$$\chi_2(P_2 \times C_n) = \begin{cases} 3, & \text{if } n \equiv 0 \pmod 3, \\ 4, & \text{otherwise} \end{cases}$$

by considering that $P_2 \times C_n$ is isomorphic to C_{2n} or two copies of C_n.

Let $G = (V, E) = P_m \times C_n$. Assume that $V = \{(i, j) | 0 \le i \le m-1,\ 0 \le j \le n-1\}$ and $E = \{\{(i, j), (i', j')\} | (i, j), (i', j') \in V,\ i' = i \pm 1,\ j' \equiv j \pm 1 \pmod n\}$. Fig. 1 shows $P_4 \times C_7$.

Shiu and Wu [23] introduced the circular palisade graph $CP(m, n)$. They also found the $L(j, k)$-number of the direct product of a path and a cycle for $2j \le k$. We denote $CP(m, n)$ by $G_{m,2n}$. Define $G_{m,n} = (V_{m,n}, E_{m,n})$ to be a graph with its vertex set $V_{m,n} = \{(i, j) | 0 \le i \le m-1,\ 0 \le j \le n-1,\ i+j \text{ is even}\}$ and edge set $E_{m,n} = \{\{(i, j), (i', j')\} | (i, j), (i', j') \in V_{m,n} | i' \equiv i \pm 1 \pmod m),\ j' \equiv j \pm 1 \pmod n\}$. Then $P_m \times C_n$ is isomorphic to $G_{m,2n}$ when n is odd by the following isomorphism: $g : P_m \times C_n \to G_{m,2n}$ such that

$$g(i, j) = \begin{cases} (i, j), & \text{if } i + j \text{ is even}, \\ (i, j + n) & \text{if } i + j \text{ is odd}. \end{cases} \tag{1}$$

Also $P_m \times C_n$ is isomorphic to $2G_{m,n}$ when n is even. Therefore from now on it is enough to consider $G_{m,2n'}$ for a positive integer n'. Fig. 2 shows $G_{4,14}$ which is isomorphic to $P_4 \times C_7$ by g. Also Fig. 3 shows $P_4 \times C_6$ and $2G_{4,6}$ which are isomorphic to each other.

Lemma 1. *Suppose that there is a 2-distance coloring $f : V_{3,n} \to [0, 4]$ of $G_{3,n}$ with five colors. For each color $r \in [0, 4]$, define $U_r = \{j | f(i, j) = r \text{ for some } i\}$. If $U_r = \{j_1 < j_2 < \cdots < j_k\}$, then $j_{s+1} - j_s$ is 3 or 4 for all $1 \le s \le k-1$ and $j_1 + n - j_k$ is 3 or 4.*