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Locally planar graphs are 5-paintable

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ABSTRACT

A graph G is locally planar if it is embedded in a surface with large edge-width. Thomassen (1993) proved that every graph embedded in a fixed surface with sufficiently large edge-width is 5-colourable. DeVos et al. (2008) strengthened this result and proved that every graph embedded in a fixed surface with sufficiently large edge-width is 5-choosable. This paper further strengthens the result to on-line list colouring and proves that every graph embedded in a fixed surface with sufficiently large edge-width is 5-paintable.

1. Introduction

Suppose G is a graph, f is a function from $V(G)$ to $N = \{0, 1, 2, \ldots\}$. An *f-list assignment* of G is a mapping L which assigns to each vertex v of G a set $L(v)$ of $f(v)$ positive integers as permissible colours. If $f(v) = k$ for all $v \in V(G)$, then an f-list assignment is called a k-list assignment. Given a list assignment L of G, an *L-colouring* of G is a proper vertex colouring $c : V(G) \to N$ of G such that $c(v) \in L(v)$ for every vertex v. If G has an L-colouring for any k-list assignment L, then we say G is *k-choosable*. The *choice number* $ch(G)$ of G is the minimum k for which G is k-choosable. List colouring of graphs was introduced independently in the 1970s by Vizing [10] and by Erdős, Rubin and Taylor [3] and has been studied extensively in the literature [9].

The on-line list colouring of a graph, first studied by Schauz [5], is a variation of list colouring. It is defined through a two-person game. At the beginning of the game, instead of assigning to each vertex v a set $L(v)$ of $f(v)$ permissible colours, each vertex v is assigned $f(v)$ tokens. In the process of the game, each token is replaced by a permissible colour, and one needs to decide right away whether or not to colour v with this permissible colour. The precise definition is as follows:

Definition 1.1. Given a graph G and a mapping $f : V(G) \to N$. The *f-painting game on G* is played by two players: Lister and Painter. Initially, all vertices are uncoloured and each vertex v has $f(v)$ tokens. In the ith step, Lister marks a non-empty subset L_i of uncoloured vertices and takes away one token from each marked vertex. Painter chooses an independent set X_i contained in L_i and colours vertices in X_i by colour i. If at the end of some step, there is an uncoloured vertex v with no tokens left, then Lister wins the game. Otherwise, at some step, all vertices are coloured and Painter wins the game.

Thus in the f-painting game, the $f(v)$ tokens at vertex v are traded, one by one, for permissible colours. Painter needs to colour v with a permissible colour, under the restriction that no colour is assigned to two adjacent vertices. Painter's goal is to colour all the vertices and Lister's goal is the opposite.

The game was called the *on-line list colouring game* in [12], as the lists of permissible colours are given on-line and the colouring is constructed on-line. Schauz's version of the game [5] was originally described as a game between *Mr. Paint and*

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Mrs. Correct, and in [1] it was called a game between Marker and Remover (coloured vertices require no further attention and may be regarded as removed).

Definition 1.2. Suppose $f : V(G) \to N$. We say G is *f-paintable* if Painter has a winning strategy in the f-painting game on G. We say G is *s-paintable* for a positive integer s if G is f-paintable for the constant function $f \equiv s$. The *paint number* $\chi_p(G)$ (also called the *paintability* and the *on-line choice number*) of G is the least integer s for which G is s-paintable.

It follows from the definition that $\chi_p(G) \geq ch(G)$ for any graph G. There are graphs for which $\chi_p(G) > ch(G)$. However, many upper bounds for the choice number of classes of graphs remain upper bounds for their paint numbers. For example, the paint number of planar graphs is at most 5 [5], the paint number of the line graph $L(G)$ of a bipartite graph G is $\triangle(G)$, and if G has an orientation in which the number of even Eulerian subgraphs differs from the number of odd Eulerian subgraphs and $f(x) = d^+(x) + 1$, then G is f-paintable [6].

Given an upper bound for the choice number of a graph, it is natural to ask whether the same upper bound holds for its paint number. In this paper, we consider paint number of locally planar graphs. Voigt [11] proved that there are planar graphs that are not 4-choosable, and Thomassen [8] proved that all planar graphs are 5-choosable. Indeed, Thomassen proved in [8] the following stronger result.

Theorem 1.3. *Assume G is a plane graph. If $f(v) = 3$ for each vertex v on the boundary of G, and $f(v) = 5$ for each interior vertex, then G is f-choosable.*

Theorem 1.3 was strengthened to the on-line list colouring version by Schauz [5], which is crucial for the proof in this paper.

Theorem 1.4 (*[5]*)**.** *Assume G is a plane graph. If $f(v) = 3$ for each vertex v on the boundary of G, $f(v) = 5$ for each interior vertex v, then G is f-paintable.*

Graphs embedded on surfaces of higher genus of course can have large chromatic number (and hence large choice number and paint number). However, if a graph G embedded in a surface S is locally planar, meaning that it does not contain short noncontractible cycles, then one can deduce similar properties as for planar graphs. The *edge-width* of a graph G embedded in a surface S is the length of a shortest cycle which is noncontractible in S. Thomassen [7] proved that graphs in a surface S with sufficiently large edge-width are 5-colourable.

Theorem 1.5 (*[7]*)**.** *For any surface S there exists a constant w such that every graph that can be embedded in S with edge-width at least w is* 5*-colourable.*

This result was strengthened by DeVos, Kawarabayashi and Mohar, who proved in [2] that every graph embedded in a fixed surface with sufficiently large edge-width is 5-choosable.

Theorem 1.6 (*[2]*)**.** *For every surface S there exists a constant w such that every graph that can be embedded in S with edge-width at least w is* 5*-choosable.*

Naturally one wonders if this result can be further strengthened to paint number. This paper answers this question in the affirmative.

Theorem 1.7. *For every surface S there exists a constant w such that every graph that can be embedded in S with edge-width at least w is 5-paintable.*

The constant w in Theorem 1.7 necessarily depends on the surface, as there are graphs of large girth (and hence large edge-width) and large chromatic number.

The proof of Theorem 1.7 is parallel to the proof of Theorem 1.6 in [2]. We shall use a structure result concerning local planar graphs proven in [2]. Nevertheless, the on-line feature of the painting game brings about some subtle difficulties.

2. Graphs of large face-width

Let S be a fixed surface and G a graph embedded in S. The *face-width* $\text{fw}(G)$ of G is the largest integer k such that every non-contractible curve in S intersects G in at least k points. It is obvious that for any graph G embedded in S, $\text{fw}(G) \leq \text{ew}(G)$.

We shall first prove the following result.

Theorem 2.1. *For every surface S there exists a constant w such that every* 5*-connected graph that can be embedded in S with face-width at least w is* 5*-paintable.*

To prove Theorem 2.1, we may assume that G is a triangulation of the surface S, because adding edges does not decrease the face-width or the connectivity of a graph.