



Cycle double covers containing certain circuits in cubic graphs having special structures

Herbert Fleischner^{a,*}, Roland Häggkvist^b

^a Institute for Information Systems, Technical University of Vienna, Favoritenstr. 9-11, A-1040 Wien, Austria

^b Department of Mathematics, Umeå University, S-90187 Umeå, Sweden

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ABSTRACT

Our point of departure is Fleischner and Häggkvist (2014, Theorem 2). We first generalize this theorem. Then we apply it to cubic graphs whose vertex set can be decomposed into two classes, one class inducing a circuit and the other class inducing a (subdivision of a) caterpillar.

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1. Introduction and preliminary discussion

All concepts not defined in this paper can be found in the textbook by Bondy and Murty [1], with the exception that we call a connected 2-regular graph a *circuit*, whereas a *cycle* is a (not necessarily connected) graph all of whose vertices have even degree.

The Cycle Double Cover Conjecture (CDCC) has found a lot of attention since the 1970s (see, e.g., the introduction of [2]), and there is even a book on Cycle Double Covers (CDCs) by C.-Q. Zhang [3]. Now, the CDCC claims that in an arbitrary bridgeless graph G there is a family of cycles (or, equivalently, a family of circuits), S such that every edge of G is covered by exactly two elements of S . In producing a CDC one is sometimes led to construct a $(1, 2)$ -circuit cover first, which is a circuit cover using every edge at least once and at most twice.

At this point of development on the CDCC, various methods have evolved to produce CDCs in various classes of graphs, but there is no *general* method in sight yet which could result in a full solution of the CDCC. Correspondingly, we focus in this paper on the following types of connected graphs.

- (i) $G = G(C_{2k}, C_{4k}, I_{2k})$ where C_{2k} and C_{4k} are chordless circuits of length $2k, 4k$ respectively, and the remaining vertices lie in I_{2k} which is an independent set of vertices each of whose elements is adjacent to two vertices of C_{4k} and to one vertex of C_{2k} ;
- (ii) $G = G(C_1, C_2, I, E_{1,2})$, where C_1 is a circuit with $d \geq 0$ chords, C_2 is an induced circuit, I is an independent set comprising the remaining vertices with $|N(v) \cap V(C_1)| = 1, |N(v) \cap V(C_2)| = 2$ for every vertex $v \in I$, and the remaining edges lie in $E_{1,2} := E(C_1, C_2)$;
- (iii) $G = G(C, T, E(C, T); x, y)$ where G is vertex-covered by the induced circuit C and the induced tree T with $C \cap T = \{x, y\}$, and the remaining edges are in $E(C, T)$.

* Corresponding author.

E-mail address: herbert.fleischner@dbai.tuwien.ac.at (H. Fleischner).

We observe that the graphs in (i) have been treated in [2], whereas some cubic graphs in (ii), (iii) are shown in the present paper to have CDCs. Note that in the cubic case, the graphs of (ii) considered in Theorem 1 can be viewed as generalizations of the graphs of (i). Consequently, our point of departure is the following theorem.

Theorem A ([2, Theorem 2]). Let $G = G(C_{2k}, C_{4k}, I_{2k})$ be a cubic graph of the type described in (i) above. Then G has a CDC S with $C_{4k} \in S$.

Theorem A serves as the basis of the following:

Theorem 1. Let $G = G(C_1, C_2, I, E_{1,2})$ be a cubic graph of the type described in (ii) above such that $|I| \equiv |E_{1,2}| \equiv 0 \pmod{2}$ and $\ell(C_1) = |I| + |E_{1,2}| + 2d$ where d is the number of chords of C_1 . Then G has a CDC S with $C_2 \in S$.

In proving Theorems 1 and 2 we make use of two essentially standard concepts which we define here to make the proofs precise.

First of all, given a vertex set T with an even number of elements in the graph G , a T -join of G is a spanning subgraph G_0 of G such that the odd vertices in G_0 are the elements in T . Note that a connected graph G has a T -join for every even set $T \subset V(G)$.

Secondly, a *caterpillar* is a tree B such that every vertex is either on a longest path $P_{\max}$ or adjacent to a vertex on $P_{\max}$. Such $P_{\max}$ is called a *spine* of B . Here we are concerned with caterpillars B of maximum degree 3 in which case B has at most 4 spines.

Theorem 2. Let $G = G(C, T, E(C, T); x, y)$ be a cubic graph of the type described in (iii) above such that

- (1) T is a caterpillar;
- (2) x and y are the ends of a spine of T ;
- (3) the components of $C - \{x, y\}$ are paths of even length;
- (4) $|\{v \in V(T) : d_T(v) = 2\}| \equiv |\{v \in V(T) : d_T(v) = 3\}| \equiv 0 \pmod{2}$.

Then G has a CDC S with $C \in S$.

2. Proofs

Proof of Theorem 1. We start from the definition of the graphs mentioned in (ii) and the condition of the length of C_1 in the statement of the theorem. These imply that the graphs under consideration result in a graph homeomorphic to a graph of type (i) when removing the edges of $E_{1,2}$ and the chords of C_1 . Thus we consider the graph G_0 as in the proof of Theorem A; i.e., G_0 is the cubic graph homeomorphic to $G' := G - V(C_1)$. In particular, G_0 contains the Hamiltonian circuit C_2^0 corresponding to C_2 ; thus G_0 has a 3-edge coloring L_r, L_b, L_y such that $E(C_2^0) = L_r \cup L_b$. Because of the definition of the graph G and the length of C_1 , it follows that $\ell(C_2^0) = 4k$ where $2k = |I|$. Next, we consider the edge-induced subgraphs $G_{r,y}, G_{b,y}$ of G_0 defined by

$$G_{r,y} := \langle L_r \cup L_y \rangle_{G_0}, \quad G_{b,y} := \langle L_b \cup L_y \rangle_{G_0}.$$

Now we construct a bipartite graph H with vertex bipartition

$$V(H) = V_{r,y} \dot{\cup} V_{b,y}$$

such that $c_{s,y} \in V_{s,y}$ corresponds bijectively to a component $C_{s,y}$ of $G_{s,y}$ for $s \in \{r, b\}$. Moreover, $c_{r,y}$ and $c_{b,y}$ are joined by $|E(C_{r,y}) \cap E(C_{b,y})|$ edges. Observe that

$$d_H(c_{r,y}) = |E(C_{r,y}) \cap L_y| \quad \text{and} \quad d_H(c_{b,y}) = |E(C_{b,y}) \cap L_y|.$$

Because of the 3-edge-coloring considered and since $\ell(C_2^0) = 4k$, it follows that $|L_y|$ is even and thus the number of odd vertices of H in $V_{r,y}$ is even, and the same holds in $V_{b,y}$. Thus $|E(H)| \equiv 0 \pmod{2}$. Moreover, the edges of L_y and thus the edges of H correspond bijectively to the vertices of I .

We note in passing that so far the proof coincides with that of Theorem A.

Now we consider $H' := S(H)$, the subdivision graph of H ; denote by $s(e)$ the subdivision vertices of H , $e \in E(H)$. Moreover, denote by $d_{1,2}(c)$ the number of edges in $E_{1,2}$ which are incident to a vertex inserted in a red (blue respectively) edge in the component of $G_{r,y}$ ($G_{b,y}$ respectively) defining $c \in V(H)$. Observe that $\sum_{c \in V(H)} d_{1,2}(c) = |E_{1,2}|$ and that H' is connected (see the proof of Theorem A).

Define

$$T := \{s(e) \in V(H') : e \in E(H)\} \dot{\cup} \{c \in V(H) : d_H(c) \not\equiv d_{1,2}(c) \pmod{2}\}.$$

We aim at finding a T -join in H' . To this end we need to prove the following.

Claim. $|T| \equiv 0 \pmod{2}$.

Next we define the following sets:

$$\begin{aligned} V_{1,2}^e &:= \{c \in V(H) : d_{1,2}(c) \equiv 0 \pmod{2}\}, & V_{1,2}^o &:= V(H) - V_{1,2}^e, \\ V_H^e &:= \{x \in V(H) : d_H(x) \equiv 0 \pmod{2}\}, & V_H^o &:= V(H) - V_H^e. \end{aligned}$$

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