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Point determining digraphs, {0, 1}-matrix partitions, and dualities in full homomorphisms

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ABSTRACT

A digraph D is point determining if for any two distinct vertices u, v there exists a vertex w which has an arc to (or from) exactly one of u, v. We prove that every point-determining digraph D contains a vertex v such that $D - v$ is also point determining. We apply this result to show that for any $\{0, 1\}$-matrix M, with k diagonal zeros and ℓ diagonal ones, the size of a minimal M-obstruction is at most $(k+1)(\ell+1)$. This is a best possible bound, and it extends the results of Sumner, and of Feder and Hell, from undirected graphs and symmetric matrices to digraphs and general matrices.

1. Introduction

We consider partitions of a digraph D into sets that satisfy certain internal constraints (the set induces an independent set or a clique), and external constraints (a sets is completely adjacent or completely non-adjacent to another set). These constraints are encoded in a $\{0, 1\}$-matrix (also called a $\{0, 1\}$-*pattern* [4]) M defined below. We assume that the digraph D has no loops. (We will allow loops, but only in a digraph that will be denoted exclusively by H.) The *in-neighbourhood* (*out-neighbourhood*) of a vertex v, denoted by $N^-(v)$ (respectively by $N^+(v)$), is the set of all vertices u in D such that $(u, v) \in A(D)$ ($(v, u) \in A(D)$). A *strong clique* of D is a set C of vertices such that for any two distinct vertices $x, y \in C$ both arcs (x, y), (y, x) are in D; and an *independent set* of D is a set I of vertices such that for any two vertices $x, y \in C$ neither pair (x, y), (y, x) is an arc of D. Let S, S' be two disjoint sets of vertices of D: we say that S is *completely adjacent to* S' (or S' is completely adjacent from S) if for any $x \in S$, $x' \in S'$, the arc (x, x') is in D; and we say that S is *completely non-adjacent to* S' (or S' is completely non-adjacent from S) if for any $x \in S$, $x' \in S'$, the pair (x, x') is not an arc of D.

Throughout this paper, M will be a $\{0, 1\}$-matrix with k diagonal 0's and ℓ diagonal 1's. For convenience we shall assume that the rows and columns of M are ordered so that the first k diagonal entries are 0, and the last ℓ diagonal entries are 1. (Thus $k + \ell$ is the size of the matrix.)

An *M-partition* of a digraph D is a partition of its vertex set $V(D)$ into parts $V_1, V_2, \ldots, V_{k+\ell}$ such that

- V_i is an independent set of D if $M(i, i) = 0$.
- V_i is a strong clique of D if $M(i, i) = 1$.
- V_i is completely non-adjacent to V_j if $M(i, j) = 0$.
- V_i is completely adjacent to V_j if $M(i, j) = 1$.

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In [3] we introduced a more general version of matrix partitions, in which matrices are allowed to have an $*$ entry implying no restriction on the corresponding set, or pair of sets. For a survey of results on M-partitions we direct the reader to [4].

A *full homomorphism* of a digraph D to a digraph H is a mapping $f : V(D) \to V(H)$ such that for distinct vertices x and y, the pair (x, y) is an arc of D if and only if $(f(x), f(y))$ is an arc of H. The following observation is obvious: let H denote the digraph whose adjacency matrix is M. (Note that H has loops if $\ell > 0$.) Then D admits an M-partition if and only if it admits a full homomorphism to H. It should be pointed out that our definition of full homomorphism (in particular the requirement that x, y be distinct) is tailored to correspond to matrix partitions as defined in [3]. The standard definition [6,7,1] does not require this distinctness; this accounts for small discrepancies between the results of this paper and that of [1]. However, when H has no loops, i.e., when $\ell = 0$, the two definitions coincide.

Undirected graphs are viewed in this paper as special cases of digraphs, i.e., each undirected edge xy is viewed as the two arcs (x, y), (y, x). For a symmetric $\{0, 1\}$-matrix M, the same definition applies to define an M-partition of an undirected graph G [3,4].

The questions investigated here have been studied for undirected graphs in [2,1], cf. [4]. It is shown in [2,1] that for any symmetric $\{0, 1\}$-matrix M (i.e., any undirected graph H with possible loops) there is a finite set $\mathcal{G}$ of graphs such that G admits an M-partition (i.e., a full homomorphism to H) if and only if it does not contain an induced subgraph isomorphic to a member of $\mathcal{G}$. This property is what [1] calls a *duality* of full homomorphisms. Alternately [4], we define a *minimal obstruction* to M-partition to be a digraph D which does not admit an M-partition, but such that for any vertex v of D, the digraph $D - v$ does admit an M-partition. Thus the results of [1,2] imply that each symmetric $\{0, 1\}$-matrix M has only finitely many minimal graph obstructions. In [2] it is shown that these minimal graph obstructions have at most $(k + 1)(\ell + 1)$ vertices, and that there are at most two minimal graph obstructions with precisely $(k + 1)(\ell + 1)$ vertices. For the purposes of this proof, the authors of [2] consider the following concept. A graph is *point determining* if distinct vertices have distinct open neighbourhoods. According to Sumner [8], each point determining graph H contains a vertex v such that $H - v$ is also point determining; the authors of [2] derived their bound by proving a refined version of Sumner's result.

For digraphs (and $\{0, 1\}$-matrices M that are not necessarily symmetric), it is still true that each M has at most a finite set of minimal digraph obstructions [1,4]. In this paper we prove that the optimal bound still applies, i.e., that it is still the case that each minimal digraph obstruction has at most $(k + 1)(\ell + 1)$ vertices. (This was conjectured in earlier versions of [4].) For this purpose we define a digraph version of point determination and prove the analogue of Sumner's result. Since undirected graphs can be viewed as symmetric digraphs, our results imply the $(k + 1)(\ell + 1)$ bound for graphs from [2], as well as the basic version of Sumner's result.

We leave open the question whether a $\{0, 1\}$-matrix M always has at most two minimal digraph obstructions with $(k+1)(\ell + 1)$ vertices; we do not have a counterexample.

In Section 2, we prove the above digraph version of Sumner's theorem, using the tools from [2]. In Section 3 we use this result to derive our $(k + 1)(\ell + 1)$ bound for the size of a minimal M-obstruction which has no (true or false) twins. In Section 4 we do the same for minimal M-obstructions that do have twins.

2. Point-determining digraphs

Let D be a digraph and let u, v, w be distinct vertices in D; we say that vertex w *distinguishes* vertices u, v in D if exactly one of u, v is in the in-neighbourhood of w, or exactly one of u, v is in the out-neighbourhood of w. We say that u, v are *twins* in D if there is no vertex that distinguishes them in D. We say that twins u, v are *true* twins if $\{u, v\}$ is a strong clique and *false* twins if $\{u, v\}$ is an independent set. We say that a digraph is *point-determining* if it does not contain a pair of false twins. Note that D has no true twins if and only if the complement of D is point-determining.

In this section we will prove the following digraph analogue to Sumner's theorem.

Theorem 1. *If D is a point-determining digraph, then there exists at least one vertex $v \in V(D)$ such that $D - v$ is point-determining.*

To prove this we will consider the notion of a triple in a point-determining digraph (cf. [2] for an analogous undirected concept). Let D be a point-determining digraph. A triple $T = (x, \{y, z\})$ of G consists of a vertex x of D, called the red vertex of T, and an unordered pair $\{y, z\}$ of vertices of D, called the green vertices of T, such that y, z are false twins in $D - x$. (Thus x is the only vertex of G that distinguishes y and z.) We begin with two lemmas.

Lemma 2. *Let D be a point-determining digraph, and let T_1 and T_2 be two triples of D. If T_1 and T_2 intersect in a vertex that is green in T_1 and red in T_2, then they intersect in another vertex that is green in T_2 and red in T_1.*

Proof. Consider two triples that share a vertex z which is red in one triple and green in the other, say triples $T_1 = (z, u, v)$ and $T_2 = (x, y, z)$. If $\{x, y\} \cap \{u, v\} = \varnothing$, then since z is the unique vertex distinguishing u and v, the vertex y does not distinguish u and v. This means that one of the vertices u, v distinguishes y and z, which contradicts the fact that $(x, \{y, z\})$ is a triple of D (*i.e.*, x is the only vertex of D distinguishing y and z). If $y \in \{u, v\}$ and $x \notin \{u, v\}$, say, $y = u$ and $v \neq x$, then v is not adjacent to $u = y$, so v is not adjacent to z, because $(x, \{y, z\})$ is a triple and $v \neq x$. The vertices $u = y$ and z are not adjacent either, as $(x, \{y, z\})$ is a triple; this contradicts the fact that $(z, \{u, v\})$ is a triple. Therefore x must be one of u, v. □

Lemma 3. *Let D be a point-determining digraph. There exists at least one vertex in D that is red in no triple of D.*