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The Maximum Independent Set Problem in Subclasses of Subcubic Graphs

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ABSTRACT

The Maximum Independent Set problem is NP-hard and remains NP-hard for graphs of maximum degree at most three (also called *subcubic graphs*). In this paper, we will study its complexity in subclasses of subcubic graphs.

Let $S_{i,j,k}$ be the graph consisting of three induced paths of lengths i, j and k, with a common initial vertex and A_q^l be the graph consisting of an induced cycle C_q $(q \geq 3)$ and an induced path with l edges having an end-vertex in common with the C_q. For example $S_{0,0,l}$ and A_4^1 are known as path of length l and banner, respectively.

Our main result is that the Maximum Independent Set problem can be solved in polynomial time in the class of subcubic, $(S_{2,j,k}, A_5^l)$-free graphs.

1. Introduction

Let us consider a finite, undirected graph G with vertex set $V(G)$ and edge set $E(G)$. A vertex subset $I \subseteq V(G)$ is called *independent* (also called *stable*) if there exists no edge between two pairwise distinct vertices $u, v \in I$. Then the Maximum Independent Set problem (MISP) asks for an independent set of maximum cardinality. By $\alpha(G)$, we denote the cardinality of such a set, which is known as *independence number* of G. $N(v)$ $(N[v])$ is the (closed) neighborhood of v for some vertex $v \in V(G)$. Using these notations, we define $N(U) = \bigcup_{v\in U} N(v) \setminus U$ $(N[U] = N(U) \cup U)$ as the (closed) neighborhood of a vertex subset $U \subseteq V(G)$. As usual, we denote by $d(u) = |N(u)|$ the degree of a vertex, by $\delta(G)$ and $\Delta(G)$ the *minimum degree* and *maximum degree* of a graph G, respectively. If the maximum degree is at most three, then we call such a graph *subcubic*. As the *diameter* of a graph, we define the greatest distance between two vertices.

For a vertex subset $U \subseteq V(G)$, we denote by $G[U]$ the graph with vertex set U and edge set $E(G) \cap \{uv : u, v \in U\}$, i.e. the graph obtained by deleting all vertices of $V(G) \setminus U$ together with their incident edges. Thus, a graph H is an *induced subgraph* of G if there exists a set $U \subseteq V(G)$ such that $G[U]$ is isomorphic to H. For any $u \in V(G)$ and $U \subseteq V(G)$, let us denote by $G - u$ and $G - U$ the graph $G[V(G) \setminus \{u\}]$ and $G[V(G) \setminus U]$, respectively.

By $\mathcal{S}$, we denote the set of all graphs of the form $S_{i,j,k}$ for $i \leq j \leq k$ where $S_{i,j,k}$ is the graph consisting of three induced paths of lengths i, j and k with a common initial vertex (see Fig. 1). Any edge of a $S_{i,j,k}$ is a *tail-edge*. By A_q^l we denote the graph consisting of an induced cycle C_q $(q \geq 3)$ and an induced path with l edges having an end-vertex in common with the C_q (see Fig. 2). All edges of the induced C_q are *cycle-edges* while the others are *tail-edges*.

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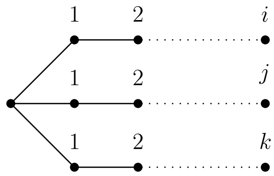

Fig. 1. $S_{i,j,k}$.

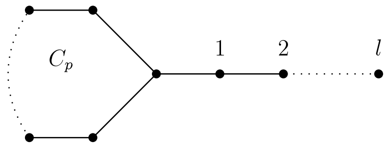

Fig. 2. A_p^l.

The MISP is well-known to be NP-hard for general graphs [9] and solvable in polynomial time for graphs of maximum degree at most two (folklore). Due to Garey et al. [6], it remains NP-hard in the class of subcubic graphs. It motivates us to study the MISP in subclasses of subcubic graphs. We define such a class in terms of forbidden induced subgraphs. Let $\mathcal{F}$ be a set of graphs. We call a graph $\mathcal{F}$-free if it does not contain an induced subgraph belonging to $\mathcal{F}$. One of the most important results in studying the MISP in graph classes defined in terms of forbidden induced subgraphs is given by Alekseev in [1].

Theorem 1.1 (*Alekseev [1]*)**.** *Let $\mathcal{F}$ be a finite set of connected graphs. Then the MISP is NP-hard in the class of $\mathcal{F}$-free graphs if $\mathcal{F} \cap \mathcal{S} = \emptyset$.*

Let us define the class of subcubic graphs in terms of forbidden induced subgraphs. Let $\mathcal{D}$ be the set of non-isomorphic connected graphs with five vertices, where at least one vertex has degree four. It is easy to see that the cardinality of this set is eleven and the intersection of $\mathcal{D}$ and $\mathcal{S}$ is empty. Because any vertex of a $\mathcal{D}$-free graph has degree at most three, we get the following corollary.

Corollary 1.2 (*Folklore*)**.** *Let $\mathcal{F}$ be a finite set of connected graphs. Then the MISP is NP-hard in the class of subcubic, $\mathcal{F}$-free graphs if $\mathcal{F} \cap \mathcal{S} = \emptyset$.*

Lozin et al. [12] conjectured the following.

Conjecture 1.3 (*Lozin et al. [12]*)**.** *Let k be fixed. Then the MISP is solvable in polynomial time for subcubic, $S_{k,k,k}$-free graphs.*

After some remarks on the current state of research in Section 2, we give some useful techniques in Section 3. Using these results we obtain in Sections 4 and 5 polynomial-time algorithms for the MISP in the classes of subcubic, $S_{1,j,k}$-free graphs and subcubic, $(S_{2,j,k}, A_5^l)$-free graphs for some fixed $j, k, l \in \mathbb{N}$, respectively.

2. Known results

The first forbidden subgraphs one can consider are paths. Let us denote a path of length l by $S_{0,0,l}$. To show that the MISP is solvable in polynomial time for subcubic, $S_{0,0,l}$-free graphs, we use a diameter technique.

Claim 2.1. *Let $d, p \in \mathbb{N}$ and $\mathcal{X}$ be a class of graphs where every graph has maximum degree at most p and diameter at most d. Then $\mathcal{X}$ is finite and the MISP is solvable in polynomial time in $\mathcal{X}$.*

Proof. If G is a graph of $\mathcal{X}$, then its order is bounded from above by $1 + p \cdot \sum_{i=0}^{d-1} (p-1)^i$. Hence, $\mathcal{X}$ is finite. Obviously, the MISP is solvable in polynomial time for any finite class of graphs. □

For any $S_{0,0,l}$-free graph the diameter is at most $l - 1$. Claim 2.1 gives us the following Lemma.

Lemma 2.2 (*Folklore*)**.** *Let l be fixed. Then the MISP is solvable in polynomial time for subcubic, $S_{0,0,l}$-free graphs.*

Knowing that the MISP is polynomially solvable in subcubic graphs without some long induced path, for example subcubic, $S_{0,0,l}$-free graphs, we may forbid the $S_{1,j,k}$ as an induced subgraph, where j, k are fixed. Sbihi [14] and Minty [13] independently proved that the MISP is solvable in polynomial time for $S_{1,1,1}$-free (also known as claw-free) graphs without a restriction on the maximum degree. Alekseev extended this result to $S_{1,1,2}$-free graphs (also known as fork-free or chair-free graphs) in [2]. To the best of our knowledge, there exists no other result in forbidding only $S_{i,j,k}$ for some $i, j, k \geq 1$. Only if we add some restrictions, for example a bounded maximum degree, results for other graph classes have been shown.