Discrete Mathematics 338 (2015) 1779–1783

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On choosability with separation of planar graphs with lists of different sizes

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ARTICLE INFO

Article history:
Received 30 September 2013
Accepted 10 January 2015
Available online 7 February 2015

Keywords:
Graph coloring
Planar graphs
Choosability with separation
List coloring

ABSTRACT

A (k, d)-*list assignment* L of a graph G is a mapping that assigns to each vertex v a list $L(v)$ of at least k colors and for any adjacent pair xy, the lists $L(x)$ and $L(y)$ share at most d colors. A graph G is (k, d)-choosable if there exists an L-coloring of G for every (k, d)-list assignment L. This concept is also known as choosability with separation.

It is known that planar graphs are $(4, 1)$-choosable but it is not known if planar graphs are $(3, 1)$-choosable. We strengthen the result that planar graphs are $(4, 1)$-choosable by allowing an independent set of vertices to have lists of size 3 instead of 4.

Our strengthening is motivated by the observation that in $(4, 1)$-list assignment, vertices of an edge have together at least 7 colors, while in $(3, 1)$-list assignment, they have only at least 5. Our setting gives at least 6 colors.

1. Introduction

Given a graph G, a *list assignment* L is a mapping assigning to each vertex $v \in V(G)$ a list of colors $L(v)$. An L-*coloring* is a vertex coloring φ such that $\varphi(v) \in L(v)$ for each vertex v and $\varphi(x) \neq \varphi(y)$ for each edge xy. A graph G is said to be k-*choosable* if there is an L-coloring for each list assignment L where $|L(v)| \geq k$ for each vertex v. The minimum such k is called the *list chromatic number* or *choice number* of G, denoted by $\chi_\ell(G)$. A graph G is said to be (k, d)-*choosable* if there is an L-coloring for each list assignment L where $|L(v)| \geq k$ for each vertex v and $|L(x) \cap L(y)| \leq d$ for each edge xy.

This concept is called *choosability with separation*, since the second parameter may force the lists of adjacent vertices to be somewhat separated. If G is (k, d)-choosable, then G is also (k', d')-choosable for all $k' \geq k$ and $d' \leq d$. A graph is (k, k)-choosable if and only if it is k-choosable. Clearly, all graphs are $(k, 0)$-choosable for $k \geq 1$. Thus, for a graph G and each $1 \leq k < \chi_\ell(G)$, there is some threshold $d \in \{0, \ldots, k-1\}$ such that G is (k, d)-choosable but not $(k, d+1)$-choosable.

The concept of choosability with separation was introduced by Kratochvíl, Tuza, and Voigt [4]. They used the following, more general definition. A graph G is (p, q, r)-*choosable*, if for every list assignment L with $|L(v)| \geq p$ for each $v \in V(G)$ and $|L(u) \cap L(v)| \leq p - r$ whenever u, v are adjacent vertices, G is q-tuple L-colorable. Since we consider only $q = 1$ in this paper, we use a simpler notation. They investigate this concept for both complete graphs and sparse graphs. The study of dense graphs was extended to complete bipartite graphs and multipartite graphs by Füredi, Kostochka, and Kumbhat [3,2].

Thomassen [6] proved that planar graphs are 5-choosable, and hence they are $(5, d)$-choosable for all d. Voigt [8] constructed a non-4-choosable planar graph, and there are also examples of non-$(4, 3)$-choosable planar graphs. Kratochvíl, Tuza, and Voigt [4] showed that all planar graphs are $(4, 1)$-choosable and asked the following question.

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http://dx.doi.org/10.1016/j.disc.2015.01.008
0012-365X/

Question 1 ([4]). Are all planar graphs (4, 2)-choosable?

Voigt [7] also constructed a non-3-choosable triangle-free planar graph. Škrekovski [5] observed that there are examples of triangle-free planar graphs that are not (3, 2)-choosable, and posed the following question.

Question 2 ([5]). Are all planar graphs (3, 1)-choosable?

Kratochvíl, Tuza and Voigt [4] proved a partial case of Question 2 by showing that every triangle-free planar graph is (3, 1)-choosable.

Choi et al. [1] proved that every planar graph without 4-cycles is (3, 1)-choosable and that every planar graph without 5-cycles and 6-cycles is (3, 1)-choosable.

In this paper we give a strengthening of the result that every planar graph is (4, 1)-choosable by allowing some vertices to have lists of size three. In a (4, 1)-list assignment L on G, for every $uv \in E(G)$ holds that $|L(u) \cup L(v)| \geq 7$. In a (3, 1)-list assignment L, for every $uv \in E(G)$ holds that $|L(u) \cup L(v)| \geq 5$. An intermediate step is to investigate the case where for every $uv \in E(G)$ holds that $|L(u) \cup L(v)| \geq 6$.

A $(*, 1)$-*list assignment* is a list assignment L where $|L(v)| \geq 1$ and $|L(u) \cap L(v)| \leq 1$ for every pair of adjacent vertices u, v.

The main result of this paper is the following theorem.

Theorem 3. *Let G be a planar graph and $I \subseteq V(G)$ be an independent set. If L is a $(*, 1)$-list assignment such that $|L(v)| \geq 3$ for every $v \in I$ and $|L(v)| \geq 4$ for every $v \in V(G) \setminus I$ then G has an L-coloring.*

The following theorem shows that it is not possible to strengthen Theorem 3 by allowing $|L(v)| \geq 2$ for every vertex $v \in V(G)$ and requiring that $|L(u) \cup L(v)| \geq 6$ for every $uv \in E(G)$.

Theorem 4. *For every k there exists a planar graph G and a $(*, 1)$-list assignment L such that $|L(v)| \geq 2$ for every $v \in V(G)$, $|L(u) \cup L(v)| \geq k$ for every $uv \in E(G)$, and G is not L-colorable.*

We first give some notation. In the next section, we prove Theorem 3 using Thomassen's precoloring extension method. In the last section we show a construction proving Theorem 4.

1.1. Notation

Given a graph G and a cycle $K \subset G$, an edge uv of G is a *chord* of K if $u, v \in V(K)$, but uv is not an edge of K. If G is a plane graph, then let $\mathrm{Int}_K(G)$ be the subgraph of G consisting of the vertices and edges drawn inside the closed disc bounded by K, and let $\mathrm{Ext}_K(G)$ be the subgraph of G obtained by removing all vertices and edges drawn inside the open disc bounded by K. In particular, $K = \mathrm{Int}_K(G) \cap \mathrm{Ext}_K(G)$. Finally, denote the characteristic function of a set S by ι_S. So $\iota_S(x) = 1$ if $x \in S$; else $\iota_S(x) = 0$.

2. Main theorem

In this section, we prove Theorem 3 by proving a slightly stronger theorem that is more amenable to induction. Observe that any list assignment satisfying the assumptions of Theorem 3 also satisfies the conditions of the following theorem.

Theorem 5. *Let G be a plane graph with outer face F and let P be a subpath of F containing at most two vertices. Let $I \subseteq V(G-P)$ be an independent set. If L is a $(*, 1)$-list assignment satisfying the following conditions:*

(i) $|L(v)| \geq 4 - \iota_I(v) - \iota_{V(F)}(v) - 2\iota_{V(P)}(v)$ *for* $v \in V(G)$,
(ii) *P is L-colorable,*
(iii) *for every $v \in I$ there is at most one $p \in N(v) \cap V(P)$ with $(L(p) \cap L(v)) \neq \emptyset$,*

then G is L-colorable.

Proof. It is easy to check that the statement of the theorem is true for graphs on at most three vertices. Let $G = (V, E)$ and L be a counterexample where $|V| + |E|$ is as small as possible. Moreover, assume that the sum of the sizes of the lists is also as small as possible subject to the previous condition. Define $L(uv) = L(u) \cap L(v)$ if $uv \in E$; else $L(uv) = \emptyset$. Since G is minimal, we have the following.

Claim 1. *The following is true:*

(1) *for every $uv \in E \setminus E(P)$, $|L(uv)| = 1$;*
(2) *for every $u \in V$, $L(u) = \bigcup_{v \in N(u)} L(uv)$; and*
(3) *for every triangle uvw such that $uv, vw, uw \in E \setminus E(P)$, $L(uv) = L(vw)$ implies $L(uv) = L(uw)$.*