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# On Wiman's theorem for graphs

Alexander Mednykh [a,b,*], Ilya Mednykh [a,c]

[a] Sobolev Institute of Mathematics, Novosibirsk State University, 630090, Novosibirsk, Russia
[b] Univerzita Mateja Bela, 97401, Banská Bystrica, Slovakia
[c] Siberian Federal University, 660041, Krasnoyarsk, Russia

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ABSTRACT

The aim of the paper is to find discrete versions of the Wiman theorem which states that the maximum possible order of an automorphism of a Riemann surface of genus $g \geq 2$ is $4g+2$. The role of a Riemann surface in this paper is played by a finite connected graph. The genus of a graph is defined as the rank of its homology group. Let $\mathbb{Z}_N$ be a cyclic group acting freely on the set of directed edges of a graph $X$ of genus $g \geq 2$. We prove that $N \leq 2g+2$. The upper bound $N = 2g+2$ is attained for any even $g$. In this case, the signature of the orbifold $X/\mathbb{Z}_N$ is $(0; 2, g+1)$, that is $X/\mathbb{Z}_N$ is a tree with two branch points of order 2 and $g+1$ respectively. Moreover, if $N < 2g+2$, then $N \leq 2g$. The upper bound $N = 2g$ is attained for any $g \geq 2$. The latter takes a place when the signature of the orbifold $X/\mathbb{Z}_N$ is $(0; 2, 2g)$.



## 0. Introduction

Klein's quartic curve, $x^3y + y^3z + z^3x = 0$, admits the group $\mathrm{PSL}_2(7)$ as its full group of conformal automorphisms. It is characterized as the curve of smallest genus realizing the upper bound $84(g-1)$ on the order of a group of conformal automorphisms of a curve of genus $g > 1$, given by Hurwitz [9] in 1893. Around the same time, Wiman [15] characterized the curves $w^2 = z^{2g+1} - 1$ and $w^2 = z(z^{2g} - 1)$, $g > 1$, as the unique curves of genus $g$ admitting cyclic automorphism groups of the largest and the second largest possible order ($4g+2$ and $4g$, respectively). The modern proof of these and similar results is contained in the paper by K. Nakagawa [13].

Over the last decade, counterparts of many theorems from the classical theory of Riemann surfaces were derived in the discrete case [1,3,5,12]. In these theorems, the finite connected graphs play the role of algebraic curves and the conformal automorphisms are replaced by harmonic ones. We say that a finite group acts harmonically on a graph if it acts freely on the set of directed edges. Following [1] we define the genus of a graph as the rank of its homology group. Then the upper bound on the order of a group acting harmonically on a graph of genus $g > 1$ is $6(g-1)$. This result was obtained by S. Corry [5].

The aim of the present paper is to find a discrete version of the Wiman theorem.

Let $\mathbb{Z}_N$ be a cyclic group acting freely on the set of directed edges of a graph $X$ of genus $g \geq 2$. We prove that $N \leq 2g+2$. The upper bound $N = 2g+2$ is attained for any even $g$. Moreover, if $N < 2g+2$, then $N \leq 2g$. The upper bound $N = 2g$ is attained for any $g \geq 2$. We describe also the signature of the quotient graphs $X/\mathbb{Z}_N$ arising in these cases. See Theorems 3 and 4 for explicit statements of the results.

* Corresponding author at: Sobolev Institute of Mathematics, Novosibirsk State University, 630090, Novosibirsk, Russia.
*E-mail addresses:* smedn@mail.ru (A. Mednykh), ilyamednykh@mail.ru (I. Mednykh).

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The basic tools to establish main results of the paper are the Riemann–Hurwitz theorem for graphs proved in [1] and the discrete version of the Harvey theorem (Theorem 2) proved in Section 2. In turn, the proof of Theorem 2 is based on the theory of harmonic morphisms [1] and the Bass–Serre uniformization theory [2] for graphs of groups. The necessary preliminary results are given in Section 1.

## 1. Basic definitions and preliminary results

### 1.1. Graphs

In this paper, by a *graph* $X$ we mean a connected multigraph. Denote by $V(X)$ the set of vertices of $X$ and by $E(X)$ the set of directed edges of $X$. Following J.-P. Serre [14] we introduce two maps $\partial_0,\ \partial_1 : E(X) \to V(X)$ (endpoints) and a fixed point free involution $e \to \bar{e}$ of $E(X)$ (reversal of orientation) such that $\partial_i\bar{e} = \partial_{1-i}e$. We will often identify $X$ with $V(X)$, writing $a \in X$ to mean $a$ is a vertex, but keeping the notation $E(X)$ for edges. We put

$$\mathrm{St}(a) = \mathrm{St}^X(a) = \partial_0^{-1}(a) = \{e \in E(X) | \partial_0 e = a\},$$

the *star* of $a$ and call $\deg(a) = |(St(a))|$ the *degree* (or *valency*) of $a$. A *morphism* of graphs $\varphi : X \to Y$ carries vertices to vertices, edges to edges, and, for $e \in E(X)$, $\varphi(\partial_i e) = \partial_i\varphi(e)$, $(i = 0, 1)$ and $\varphi(\bar{e}) = \bar{\varphi}(e)$. For $a \in X$ we then have the local map

$$\varphi_a : \mathrm{St}^X(a) \to \mathrm{St}^Y(\varphi(a)).$$

A map $\varphi$ is *locally bijective* if $\varphi_a$ is bijective for all $a \in X$. We call $\varphi$ a *covering* if $\varphi$ is surjective and locally bijective. A bijective morphism is called an *isomorphism*, and an isomorphism $\varphi : X \to X$ is called an *automorphism.*

### 1.2. Harmonic morphisms and harmonic actions

Let $X,\ Y$ be graphs. Let $\varphi : X \to Y$ be a morphism of graphs. We now come to one of the key definitions in this paper.

A morphism $\varphi : X \to Y$ is said to be *harmonic* (alternatively it is called *branched covering, quasi-covering* or *vertically holomorphic map*) if, for all $x \in V(X),\ y \in V(Y)$ such that $y = \varphi(x)$, the quantity

$$|e \in E(X) : x = \partial_0 e,\ \varphi(e) = e'|$$

is the same for all edges $e' \in E(Y)$ such that $y = \partial_0 e'$.

One can check directly from the definition that the composition of two harmonic morphisms is again harmonic. Therefore the class of all graphs, together with the harmonic morphisms between them, forms a category. It is important to say that the definition of a harmonic morphism given in [1] is slightly more general. We note also that an arbitrary covering of graphs is a harmonic morphism.

Let $\varphi : X \to Y$ be harmonic and let $x \in V(X)$. We define the *multiplicity* of $\varphi$ at $x$ by

$$m_\varphi(x) = |e \in E(X) : x = \partial_0 e,\ \varphi(e) = e'| \qquad (1)$$

for any edge $e' \in E(X)$ such that $\varphi(x) = \partial_0 e'$. By the definition of a harmonic morphism, $m_\varphi(x)$ is independent of the choice of $e'$.

If $\deg(x)$ denotes the degree of a vertex $x$, we have the following basic formula relating the degrees and multiplicity:

$$\deg(x) = \deg(\varphi(x))m_\varphi(x). \qquad (2)$$

We define the degree of a harmonic morphism $\varphi : X \to Y$ by the formula

$$\deg(\varphi) := |e \in E(X) : \varphi(e) = e'| \qquad (3)$$

for any edge $e' \in E(Y)$. By [1, Lemma 2.2] the right-hand side of (3) does not depend on the choice of $e'$ and therefore $\deg(\varphi)$ is well defined.

Let $G < \mathrm{Aut}\,(X)$ be a group of automorphisms of a graph $X$. An edge $e \in E(X)$ is called *invertible* if there is an automorphism in group $G$ sending $e$ to $\bar{e}$. We say that the group $G$ acts *harmonically* on a graph $X$ if $G$ acts freely on the set $E(X)$ of directed edges of $X$. If, additionally, $G$ acts without invertible edges we say that $G$ acts *purely harmonically* on $X$. In the latter case the quotient graph $X/G$ is well defined. The vertices and the edges of $X/G$ are formed by orbits $Gu,\ u \in V(X)$ and $Ge,\ e \in E(X)$ respectively. The image of an edge $e$ with endpoints $\{u,\ v\}$ under the canonical map $X \to X/G$ is the edge $Ge$ with endpoints $\{Gu, Gv\}$.

The following observation made by Scott Corry [6] plays a crucial role.

**Observation 1.** *Suppose that a group $G$ acts purely harmonically on a graph $X$. Then the quotient graph $X/G$ is well defined and the canonical projection $X \to X/G$ is a harmonic morphism.*

Let $G$ be a finite group acting purely harmonically on a graph $X$. For every $\tilde{v} \in V(X)$ denote by $G_{\tilde{v}}$ the stabilizer of $\tilde{v}$ in the group $G$ and by $|G_{\tilde{v}}|$ the order of the stabilizer. Then to each vertex $v \in V(X/G)$ we prescribe the number $m_v = |G_{\tilde{v}}|$, where $\tilde{v} \in \varphi^{-1}(v)$. Since $G$ acts transitively on each fiber of $\varphi$, the numbers $m_v$ are defined correctly.